MA30041: Metric Spaces

Exercise Sheet 5: Open and closed sets

Please hand solutions in at the lecture on Monday 3rd November.

1.) For subsets A_1, A_2 of a metric space (X, d) we define the *distance* between A_1 and A_2 by

$$\operatorname{dist}(A_1, A_2) = \begin{cases} \infty & \text{if } A_1 = \varnothing \text{ or } A_2 = \varnothing, \\ \inf\{d(x, y) \mid x \in A_1, y \in A_2\} & \text{if } A_1 \neq \varnothing \neq A_2. \end{cases}$$

Obviously, we have $\operatorname{dist}(\{x\}, \{y\}) = d(x, y)$ for all $x, y \in X$. If one of the sets consists of a single point, we use the notation $\operatorname{dist}(x, A)$ instead of $\operatorname{dist}(\{x\}, A)$. Let $A_1, A_2, A_3 \subset X$. Show:

(i) $\operatorname{dist}(A_1, A_2) = \operatorname{dist}(A_2, A_1)$.

(ii) If $A_1 \subset A_2$, then $\operatorname{dist}(A_2, A_3) \leq \operatorname{dist}(A_1, A_3)$.

(iii) If $A_1 \cap A_2 \neq \varnothing$, then $\operatorname{dist}(A_1, A_2) = 0$.

(iv) $\operatorname{cl} A_1 = \{x \in X \mid \operatorname{dist}(x, A_1) = 0\}$.

(v) $\operatorname{dist}(A_1, A_2) = \operatorname{dist}(\operatorname{cl} A_1, A_2)$.

(vi) We say that A_1 and A_2 are *adjacent* if $\operatorname{dist}(A_1, A_2) = 0$. Show that being adjacent is not an equivalence relation.

2.) Let A_1, A_2 be subsets of a metric space (X, d). Show

(i) $\operatorname{diam} A_1 \;=\; \operatorname{diam}(\operatorname{cl} A_1)$.

(ii) $\operatorname{int} A_1 \cup \operatorname{int} A_2 \;\subset\; \operatorname{int}(A_1 \cup A_2)$.

(iii) $\operatorname{cl}(A_1 \setminus A_2) \;\subset\; \operatorname{cl} A_1 \setminus \operatorname{int} A_2$.

(iv) $\partial(A_1 \cup A_2) \;\subset\; (\partial A_1 \setminus \operatorname{int} A_2) \cup (\partial A_2 \setminus \operatorname{int} A_1)$.

(v) In the cases (ii)–(iv), show that "$\subset$" can in general *not* be replaced by "$=$".

3.) Consider the metric space $(C[0, 1], d_{\max})$.

(i) Let $U = \{f \in C[0, 1] \mid f(0) = 0\}$. Is U closed?

(ii) Let $A = \{f \in C[0, 1] \mid f(1) > 0\}$. What is ∂A?

(iii) We say that a function $f : [0, 1] \to \mathbb{R}$ is *monotonically increasing* if $f(x) \leq f(y)$ whenever $x < y$. Let $V = \{f \in C[0, 1] \mid f \text{ is monotonically increasing}\}$. What is ∂V?

Please turn over!

4.) In this question, $\mathbb{R}^2$ is always equipped with the Euclidean metric d_E.

(i) Which of the following sets are open respectively closed? Are they bounded or even a ball?

- $A = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ in $\mathbb{R}^2$.
- The square $A = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1\}$ in $\mathbb{R}^2$.
- $A = \{1, 2\}$ in $X = \{1, 2, 3\}$ equipped with the discrete metric.

(ii) Find the closure, interior and boundary of the following sets in $\mathbb{R}^2$:

- The closed unit ball $\overline{B}_1((0, 0))$.
- The graph of $y = \sin\left(\frac{1}{x}\right)$ for $x > 0$.

(iii) Find the boundaries of the following sets:

- The set $[0, 1)$ as a subset of the metric space $[0, 1]$ with the usual metric.
- The subset $A = \mathbb{R} \times \{0\}$ (the "x-axis") in $\mathbb{R}^2$.

5.) Let $\mathbb{Q} = \{q_k \mid k \in \mathbb{N}\}$ be an enumeration of the rational numbers. Let $A = \bigcup_{k \in \mathbb{N}} (q_k - 2^{-k}, q_k + 2^{-k})$. Show that the boundary of A, as a subset of $\mathbb{R}$ with the usual metric, is $\mathbb{R} \setminus A = A^c$. What makes this example "surprising"?

6.) We equip $\mathbb{Z}$ with the so-called 2-*adic metric* d given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-k} & \text{where } k \text{ is the largest power of 2 which divides } |x - y|. \end{cases}$$

Thus, for example, an even number has distance at most $\frac{1}{2}$ from 0, while any odd number has distance 1 from 0. The integers divisible by 4 have distance at most $\frac{1}{4}$ from 0 etc.

(i) Check the triangle inequality.

(ii) Show: Any ball is a clopen set.

(iii) Show: Two balls $B_r(x)$, $B_{r'}(y)$ are either disjoint (i.e., $B_r(x) \cap B_{r'}(y) = \varnothing$) or one is contained in the other (i.e., $B_r(x) \subset B_{r'}(y)$ or $B_{r'}(y) \subset B_r(x)$).

(iv) Show: $(\mathbb{Z}, d)$ is not complete.
Hint: Consider the sequence $\left(\sum_{k=1}^{n} 2^k\right)$.