MA30041: Metric Spaces

Exercise Sheet 4: Open balls and open sets

Please hand solutions in at the lecture on Monday 27th October.

1.) Consider the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the product metrics d_1, d_2 respectively d_∞.

Show: A subset $A \subset \mathbb{R}^n$ is open w.r.t. any one of these metrics iff it is open w.r.t. any other.

2.) (i) Consider $(C[0,1], d_{\max})$ and let $f \equiv 0$ (i.e., $f(x) = 0 \ \forall x \in [0,1]$). Describe the open unit ball $B_1^{(d_{\max})}(f)$.

(ii) Consider $(C[0,1], d_R)$ and let $f \equiv 0$. What is the open unit ball $B_1^{(d_R)}(f)$ now?

(iii) Does one of the balls $B_1^{(d_{\max})}(f)$, $B_1^{(d_R)}(f)$ contain the other?

3.) (i) Consider the X-prickly hedgehog (see Exercise sheet 1 Question 3). Describe its open balls.

(ii) Let (X, d_D) be a discrete metric space. Describe the open and closed balls and conclude that all balls are *clopen* sets, i.e., open and closed. Is the closure of an open ball always a closed ball of the same radius?

4.) Let $G \subset \mathbb{R}$ be open.

(i) For each $x \in G$, let I_x be the union of all open intervals contained in G and containing x. Show that I_x is an open (possibly unbounded) interval.

(ii) For $x \neq y \in G$, show that $I_x = I_y$ or $I_x \cap I_y = \varnothing$.

(iii) Conclude that G is a union of countably many, pairwise disjoint open intervals.
Hint: Any open interval in $\mathbb{R}$ contains a rational number.

(iv) What might be an analogous statement in $\mathbb{R}^n$ (no proof required, just guess)?

5.) Convince yourself that the following statements are true:

(i) A subset G in a metric space (X, d) is open iff it is the union of all open balls contained in G.

(ii) A subset G is open iff $G = \operatorname{int} G$.

(iii) For any subset A of a metric space (X, d) we have $A^{\circ\circ} = A^\circ$ where $A^{\circ\circ}$ denotes $(A^\circ)^\circ$ (i.e., $\operatorname{int}(\operatorname{int} A)$),

Please turn over!

6.) We continue to explore the metrics $d_{\max}$ and d_R on $C[a,b]$ where $a < b$ (see Exercise sheet 3 Question 5).
Recall the following definition from (M11):

> A sequence $(f_n) \subset C[a,b]$ converges *uniformly* to $f : [a,b] \to \mathbb{R}$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall m \geq N$ and $\forall t \in [a,b]$ we have $|f_m(t) - f(t)| < \varepsilon$.

Check which of the following statements hold:

(i) "*A sequence* (f_n) *in* $(C[a,b], d_{max})$ *is convergent, if it converges uniformly.*"

(ii) "*A sequence* (f_n) *in* $(C[a,b], d_R)$ *is convergent, if it converges uniformly.*"

(iii) "*A sequence* (f_n) *converges uniformly, if it converges in* $(C[a,b], d_{max})$*.*"

(iv) "*A sequence* (f_n) *converges uniformly, if it converges in* $(C[a,b], d_R)$*.*"