MA30041: Metric Spaces

Exercise Sheet 3: Completions

Please hand solutions in at the lecture on Monday 20th October.

1.) Let (X, d) be a complete metric space and let (x_n) be a sequence in X such that there is a $\theta \in (0, 1)$ with $d(x_{n+2}, x_{n+1}) \leq \theta \cdot d(x_{n+1}, x_n)$ for $n \in \mathbb{N}$. Show that (x_n) is convergent.

2.) Show: If d and d' are uniformly equivalent metrics on X, then (x_n) is a Cauchy sequence w.r.t. d iff (x_n) is a Cauchy sequence w.r.t. d'.

3.) We say that a metric space (X, d) is *finite* if X is a finite set.

(i) Show: A finite metric space (X, d) is uniformly equivalent to (X, d_D) where d_D denotes the discrete metric.

(ii) Show: Any discrete finite metric space (X, d_D) is complete.

You might find the following terminology useful:
We say that a sequence $(x_n) \subset X$ is *eventually constant* if there exists an $x \in X$ and an $N \in \mathbb{N}$ s.t. $x_n = x \ \forall n \geq N$.

(iii) Deduce that any finite metric space (X, d) is complete.

4.) In this question we explore $\mathbb{N}$ equipped with usual Euclidean metric $d_E(x, y) = |x - y|$, the discrete metric d_D and the inverse metric $d_{\text{inv}}(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.

(i) Recall the terminology "*eventually constant*" (see previous question) and show: d_E, d_D, d_{inv} are equivalent, but no two of them are uniformly equivalent.

(ii) Show: $(\mathbb{N}, d_E)$ and $(\mathbb{N}, d_D)$ are complete metric spaces, while $(\mathbb{N}, d_{\text{inv}})$ is incomplete.

(iii) Complete $(\mathbb{N}, d_{\text{inv}})$.

Please turn over!

5.) We continue to explore the metrics $d_{\max}$ and d_R on $C[a,b]$ where $a < b$ (see Exercise sheet 2 Question 3).
Recall the following definition from (M11):

> A sequence $(f_n) \subset C[a,b]$ converges *pointwise* to $f : [a,b] \to \mathbb{R}$ if $\forall t \in [a,b]$ and $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall m \geq N$ we have $|f_m(t) - f(t)| < \varepsilon$.

Check which of the following statements hold:

(i) "*A sequence* (f_n) *in* $(C[a,b], d_{max})$ *is convergent, if it converges pointwise.*"

(ii) "*A sequence* (f_n) *in* $(C[a,b], d_R)$ *is convergent, if it converges pointwise.*"

(iii) "*A sequence* (f_n) *converges pointwise, if it converges in* $(C[a,b], d_{max})$*.*"

(iv) "*A sequence* (f_n) *converges pointwise, if it converges in* $(C[a,b], d_R)$*.*"