MA30041: Metric Spaces

Exercise Sheet 2: Pseudometric spaces & convergent sequences

Please hand solutions in at the lecture on Monday 13th October.

1.) (i) Let X be a set and let d be a pseudometric on X. Let $\phi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function with $\phi(0) = 0$ and such that, for all $a, b \in [0, +\infty)$,

$$\phi(a + b) \leq \phi(a) + \phi(b). \tag{1}$$

Show: $\phi \circ d$ is also a pseudometric on X.

Remarks: Given that $\phi : [0, +\infty) \to [0, +\infty)$ satisfies $\phi(0) = 0$, a sufficient condition for (1) is that ϕ is *concave*, i.e.,

$$\phi\left(tx + (1 - t)y\right) \geq t \cdot \phi(x) + (1 - t) \cdot \phi(y)$$

for all x, y and $0 \leq t \leq 1$. If ϕ is continuous on $[0, +\infty)$ and twice-differentiable on $(0, +\infty)$, then ϕ is concave iff $\phi''(x) \leq 0$ for all $x \in (0, +\infty)$.

(ii) Use the previous result to justify: If (X, d) is a metric space and $\alpha > 0$, then

- $(x, y) \mapsto \min\{d(x, y), \alpha\}$,
- $(x, y) \mapsto \frac{d(x,y)}{1+d(x,y)}$ and
- $(x, y) \mapsto \sqrt{d(x, y)}$

are metric functions on X.

2.) Let X be any set and let $d : X \times X \to \mathbb{R}$ be a pseudometric. For $x, y \in X$, define $x \sim y$ iff $d(x, y) = 0$.

(i) Show that $\sim$ is an equivalence relation on X.

(ii) For $x \in X$, let $[x]$ denote its equivalence class with respect to $\sim$, and let $X/\sim$ denote the collection of all $[x]$ with $x \in X$. Show that

$$(X/\sim) \times (X/\sim) \to \mathbb{R}, \quad ([x], [y]) \mapsto d(x, y)$$

defines a metric on $X/\sim$.

(iii) Let $X = \mathbb{R}^2$ and $d\left((x_1, x_2), (y_1, y_2)\right) = ||x_1| - |y_1||$. With which metric space can we identify $X/\sim$ in this case?

Please turn over!

3.) On $C[a,b]$ we have two metrics:

- the maximum metric $d_{\max}(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$, and
- the metric d_R given by $d_R(f,g) = \int_a^b |f(t) - g(t)| \,\mathrm{d}t$ (see Exercise Sheet 1 Question 4).

Show that these metrics are not equivalent.

4.) Suppose that $x_n \to x$ and $y_n \to y$ in a metric space (X,d) as $n \to \infty$. Show that $d(x_n, y_n) \to d(x,y)$ (in $\mathbb{R}$ with the standard metric) as $n \to \infty$.

5.) Proof Proposition II.4: In a metric space every convergent sequence is bounded.

6.) Construct a metric on $\mathbb{R}$ in which the sequence $\left(\frac{1}{n}\right)$ of inverses of natural numbers converges to a limit other than 0.