MA30041: Metric Spaces

On the Remarks to Exercise Sheet 2 Question 1 (Not Examinable!)

Here are the proofs for the claims made.

- *Let $\phi : [0, +\infty) \to [0, \infty)$ satisfy $\phi(0) = 0$. Then ϕ being concave is a suffient condition for $\phi(a+b) \le \phi(a) + \phi(b)$ (where $a, b \in [0, +\infty)$).*
  Recall the definition of a concave function:
  $$\phi(t\,x + (1-t)\,y) \ge t \cdot \phi(x) + (1-t) \cdot \phi(y) \quad \forall x, y \text{ and } 0 \le t \le 1.$$
  Clearly, $\phi(a+b) \le \phi(a) + \phi(b)$ holds if either a or b equals 0 (since $\phi(0) = 0$). Therefore, we can assume $0 < a, b < a + b$.
  In the definiting inequality for concavity, set $x = 0$ and $y = a + b$.
  - For $t = \frac{a}{a+b}$ we get: $\phi(b) \ge \frac{a}{a+b} \cdot \phi(a+b)$.
  - For $t = \frac{b}{a+b}$ we get: $\phi(a) \ge \frac{b}{a+b} \cdot \phi(a+b)$.

  Adding these two inequalities establishes the claim. □

- We show: *Let ϕ be differentiable. Then ϕ is concave iff $\phi(x) \le \phi(y) + \phi'(y)\,(x-y)$ $\forall x, y$.*
  "⇒": Define $\Psi(t) = \phi(y) + t \cdot (\phi(x) - \phi(y)) - \phi\,(y + t \cdot (x-y))$. Then we have $\Psi(0) = 0$ and – by the concavity of ϕ – also $\Psi(t) \le 0$. Differentiating Ψ yields $\Psi'(t) = \phi(x) - \phi(y) - \phi'\,(y + t \cdot (x-y)) \cdot (x-y)$. Since $\Psi(0) = 0$ and $\Psi(t) \le 0$, we know that $\Psi'(0) \le 0$. But $\Psi'(0) = \phi(x) - \phi(y) - \phi'(y)\,(x-y)$, and the result follows.
  "⇐": Set $z = y + t \cdot (x-y)$, then the inequality on the right hand yields for x and z respectively y and z the following:
  $$\phi(x) \le \phi(z) + \phi'(z)\,(x-z) \quad \text{and} \quad \phi(y) \le \phi(z) + \phi'(z)\,(y-z).$$
  Multiplying the first inequality by t and the second by $(1-t)$ and then adding them yields $t \cdot \phi(x) + (1-t) \cdot \phi(y) \le \phi(z)$. □

- *Let ϕ be twice differentiable. Then ϕ is concave iff $\phi''(x) \le 0$ for all x.*
  "⇒": Using the previous result, we have:
  $$\phi(x) - \phi(y) \le \phi'(y)\,(x-y) \quad \text{and} \quad \phi(y) - \phi(x) \le \phi'(x)\,(y-x).$$
  Adding them yields $0 \le (\phi'(y) - \phi'(x)) \cdot (x-y)$. But this last inequality, together with the definition $\phi''(x) = \lim_{h\to 0} \frac{\phi'(x+h)-\phi'(x)}{h}$, yields $\phi''(x) \le 0$.
  "⇐": By Taylor's Theorem we have
  $$\phi(x) = \phi(y) + \phi'(y) \cdot (x-y) + \frac{1}{2}\phi''\,(y + t\,(x-y)) \cdot (x-y)^2 \text{ for some } t \in [0,1].$$
  Noting that the third term on the right hand side is nonpositive, we get $\phi(x) \le \phi(y) + \phi'(y) \cdot (x-y)$. Now the previous result establishes the claim. □