MA30041: Metric Spaces

Exercise Sheet 1: Examples of Metric Spaces

Please hand solutions in at the lecture on Monday 6th October.

1.) Let (X, d) be a metric space.

(i) Show that for all $x, y, z \in X$, $|d(x, z) - d(z, y)| \leq d(x, y)$.

(ii) Show that if $x_1, \ldots, x_{n+1} \in X$, then $d(x_1, x_{n+1}) \leq \sum_{i=1}^{n} d(x_i, x_{i+1})$.

2.) In each of the following cases, state with (careful) justification if (X, d) is a metric space.

(i) $X = \mathbb{R}^2$ equipped with $d((x_1, x_2), (y_1, y_2)) = |x_2 - y_2|$.

(ii) $X = \mathbb{Q}$ equipped with $d(x, y) = (x - y)^3$.

(iii) $X = \mathbb{C}$ equipped with $d(z_1, z_2) = \min\{|z_1| + |z_2|, |z_1 - 1| + |z_2 - 1|\}$ if $z_1 \neq z_2$, $d(z, z) = 0$ otherwise.

(iv) $X = \mathbb{R}$ equipped with $d(x, y) = |x^2 - y^2|$.

(v) $X = \mathbb{R} \setminus \mathbb{Q}$ equipped with $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.

3.) Let X be a set, $Y = \{0\} \cup (X \times (0, 1])$. For $y, y' \in Y$ we set:

$$d(y, y') = \begin{cases} 0 & \text{if } y = 0 = y', \\ r & \text{if } y = 0, y' = (x, r), \\ r & \text{if } y = (x, r), y' = 0, \\ |r - r'| & \text{if } y = (x, r), y' = (x, r'), \\ r + r' & \text{if } y = (x, r), y' = (x', r'), \text{ where } x \neq x'. \end{cases} \tag{1}$$

(i) Show: d is a metric on Y.

(ii) The metric space (Y, d) is called the X*-prickly hedgehog* (or, if you are from America or Australia, the X*-prickly porcupine*).
Make a suggestive sketch of Y that justifies this name. What is the meaning of the metric?
Note: Sometimes d is also called the *(French/British) railway metric*.

Please turn over!

4.) Let $a, b \in \mathbb{R}$, $a \leq b$ and set

$$\begin{aligned} R[a,b] &= \{ f \mid f : [a,b] \to \mathbb{R} \text{ is Riemann-integrable } \}, \\ C[a,b] &= \{ f \mid f : [a,b] \to \mathbb{R} \text{ is continuous } \}. \end{aligned}$$

Then one has $C[a,b] \subset R[a,b]$ (M11!) and

$$d_R(f,g) = \int_a^b |f(t) - g(t)| \, \mathrm{d}t$$

defines a map $d_{\mathrm{R}} : R[a,b] \times R[a,b] \to \mathbb{R}$ (again M11!).

(i) Please check which of the defining conditions (MS1)–(MS4) of a metric hold for d_{R} and which do not hold. Justify your claims.

(ii) Let $d'_{\mathrm{R}} : C[a,b] \times C[a,b] \to \mathbb{R}$ be the restriction of d_{R} to $C[a,b] \times C[a,b]$. Show: $(C[a,b], d'_{\mathrm{R}})$ is a metric space.

5.) *will be on next week's exercise sheet*

6.) Is the distance one travels a metric, for example with the car/bus between houses/bus stops in Bath?