MA30041: Metric Spaces

EXERCISE SHEET 10: CONNECTEDNESS

Solutions will be available from Monday 8th December.

1.) Let (X, d), $(Y, \tilde{d})$ be metric spaces and equip the (Cartesian) product $X \times Y$ with any product metric, e.g., with the maximum metric d_∞ given by $d_\infty((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), \tilde{d}(x_2, y_2)\}$.

   (i) Show: If (X, d), $(Y, \tilde{d})$ are totally bounded, then $(X \times Y, d_\infty)$ is also totally bounded.

   (ii) Show: If (X, d), $(Y, \tilde{d})$ are (sequentially) compact, then $(X \times Y, d_\infty)$ is also (sequentially) compact.

   (iii) Show: If (X, d), $(Y, \tilde{d})$ are connected, then $(X \times Y, d_\infty)$ is also connected.

   (iv) Show: If (X, d), $(Y, \tilde{d})$ are path connected, then $(X \times Y, d_\infty)$ is also path connected.

2.) Consider the following subsets of $\mathbb{R}^n$ endowed with the usual Euclidean metric d_E. Which of the sets are compact and which are connected? Justify your answers.

   (i) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z \neq 0\}$

   (ii) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, x^2 + y^2 \neq 0\}$

   (iii) The graph of $f : (0, \infty) \to (0, \infty)$, $x \mapsto \frac{1}{x}$ in $\mathbb{R}^2$.

   (iv) $[0, 1] \cup [2, 3] \subset \mathbb{R}$.

   (v) $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \subset \mathbb{R}$.

3.) Which of the following metric subspaces of $\mathbb{R}^2$ (equipped with the Euclidean metric) is connected? Justify your answer.

   (i) $X = \mathbb{R} \times (\mathbb{R} \setminus \mathbb{Q})$

   (ii) $Y = (\mathbb{R} \times (\mathbb{R} \setminus \mathbb{Q})) \cup ((\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R})$

4.) Show: $C[a, b]$ equipped with the uniform metric $d_{\max}$ is connected.
Is $C[a, b]$ also connected if we use the metric given by $d_R(f, g) = \int_a^b |f(t) - g(t)| \, dt$ (see Exercise sheet 1 Question 4)?
Hint: Show that $C[a, b]$ is *convex*.

Please turn over!

5.) Prove Theorem VII.8: *A connected open subset of* $\mathbb{R}^n$ *is path connected.*
Hint: Let U be a connected open subset of $\mathbb{R}^n$. Show that, given any $x \in U$, the *path connected component* of x defined as $U_x = \{y \in U | \exists \text{ path in } U \text{ joining } x \text{ and } y\}$ is open. Then conclude that $U_x = U$.

Some comments regarding revision so far

- "Please can we have a revision session before Christmas, maybe in the Tuesday or Friday session. In the lecture, please can you give an overview of the unit, with the main results / definitions, and maybe a couple of examples?"
- "Perhaps we could email you our problems in advance of the session?"
- "I would like you to give us some revision lectures. I think it would be better if you can start one revision lecture before the holiday, and also have another revision lecture during the revision week."

Cannibals at Christmas:
Fleeing from the dark and cold season in Britain and to test their knowledge of the jungle river metric in practice, a group of 31 math students from Bath go on holidays at christmas. However, at one of their boat trips they are caught by cannibals. Pointing out the past large amounts of foreign aid (fortunately, the cannibals have not heard anything about the credit crunch yet and thus the near future of foreign aid), the cannibals agree to the following deal: All prisoners are lined up, one behind the other. Each person gets a nice (it is christmas!) hat, either a blue, red or green one. However, she/he does not know the colour of her/his hat, he/she only can see the hats of the persons in front of him/her. Then, every person, beginning with the one at the end of the line, calls out a colour. If it matches the colour of his/her hat, the person is saved.

How many of the 31 students will show up at the final exam for M41? The (first) person to email the best solution (i.e., a strategy that saves the most students) will get GBP 20.– provided the third "challenging" problem is still unsolved after the 9th January.