## MA30041: Metric Spaces

## Self-Assessment Sheet 6: Continuous functions

1.) Why is the function $f: \mathbb{R} \rightarrow \mathbb{R}^{2}, x \mapsto(\cos (x), \sin (x))$ continuous?

For a solution, click on the the following space:
The functions $x \rightarrow \cos (x)$ and $x \rightarrow \sin (x)$ are continuous from $R$ to $R$ (even uniformly continuous), thus $x \rightarrow\left(\cos (x), \sin (x)\right.$ ) is continuous from $R$ to $R^{2}$ (even uniformly continuous) by Exercise sheet 6 Question 5.
2.) Given an example of a continuous function $f$ from a metric space $(X, d)$ to $\left(Y, d^{\prime}\right)$ and a closed bounded subset $A \subset X$ s.t. $f(A)$ is neither closed nor bounded in $\left(Y, d^{\prime}\right)$.
For a solution, click on the the following space:
E.g., take $A=X=(0,1)$ and $Y=R$ with the usual metric and $f(x)=1 / x$. Then, $X$ is closed (and open) and bounded, but $f(X)=(1,00)$ is neither bounded nor closed in $R$.
3.) Recall the following definition from (M11):

A sequence $\left(f_{n}\right) \subset C[a, b]$ converges pointwise to $f:[a, b] \rightarrow \mathbb{R}$ if $\forall t \in[a, b]$ and $\forall \varepsilon>0 \exists N \in \mathbb{N}$ s.t. $\forall m \geq N$ we have $\left|f_{m}(t)-f(t)\right|<\varepsilon$.

What, by analogy, must be the corresponding definition for a $f: X \rightarrow Y$ from a metric space $(X, d)$ to $\left(Y, d^{\prime}\right)$.
For a solution, click on the the following space:
Let $f: X->Y$ and $f n$ : $X->Y(n=1,2,3, \ldots)$ be given. We say that ( fn ) converges pointwise on $X$ to $f$ if $d^{\prime}(f n(t), f(t)) \rightarrow 0$ as $n->$ oo for all $t$ in $X$. Note that the metric on $X$ plays no role here!
4.) Recall the following definition from (M11):

A sequence $\left(f_{n}\right) \subset C[a, b]$ converges uniformly to $f:[a, b] \rightarrow \mathbb{R}$ if $\forall \varepsilon>0$ $\exists N \in \mathbb{N}$ s.t. $\forall m \geq N$ and $\forall t \in[a, b]$ we have $\left|f_{m}(t)-f(t)\right|<\varepsilon$.

What, by analogy, must be the corresponding definition for a $f: X \rightarrow Y$ from a metric space $(X, d)$ to $\left(Y, d^{\prime}\right)$.
For a solution, click on the the following space:
Let $f: X->Y$ and $f n: X->Y(n=1,2,3, \ldots)$ be given. We say that ( $f n$ ) converges uniformly on $X$ to $f$ if, for every $\mathrm{e}>0$, there exists an $\mathrm{N}=\mathrm{N}(\mathrm{e})$ (depending only on e) s.t. $\mathrm{d}^{\prime}(\mathrm{fn}(\mathrm{t}), \mathrm{f}(\mathrm{t})$ ) <e for all $\mathrm{n}=>\mathrm{N}$ and all t in X . Again, note that the metric on X plays no role here!

