## MA30041: Metric Spaces

## Self-Assessment Sheet 3: Cauchy Sequences \& Completions

1.) (i) Show: A sequence $\left(x_{n}\right)$ in a metric space $(X, d)$ is a Cauchy sequence iff $\forall \varepsilon>0$ exists $N \in \mathbb{N}$ s.t. $d\left(x_{m}, x_{N}\right)<\varepsilon$ for all $m \geq N$.
For a solution, click on the the following spaces:
The only difference to the usual definition of a Cauchy sequence is that there the last part reads $\mathrm{d}(\mathrm{xm}, \mathrm{xn})<\mathrm{e}$ for all $\mathrm{m}, \mathrm{n}>=\mathrm{N}$. So, setting $\mathrm{n}=\mathrm{N}$ establishes this
" $\Rightarrow$ ": direction.

We use the triangle inequality to get $\mathrm{d}(\mathrm{xn}, \mathrm{xm})<=\mathrm{d}(\mathrm{xn}, \mathrm{xN})+\mathrm{d}(\mathrm{xN}, \mathrm{xm})<2$ e for all " $\Leftarrow$ ": $\mathrm{n}, \mathrm{m}>=\mathrm{N}$. This establishes the claim.
(ii) Show: A sequence $\left(x_{n}\right)$ in a metric space $(X, d)$ is a Cauchy sequence iff

$$
\lim _{N \rightarrow \infty} \operatorname{diam}\left\{x_{m} \mid m \geq N\right\}=0
$$

For a solution, click on the the following spaces:
Since ( xn ) is a Cauchy sequence, there is, for every $\mathrm{e}>0$, an N s.t. $\mathrm{d}\left(\mathrm{x} \_\mathrm{m}, \mathrm{x} \_\mathrm{n}\right)<$
$" \Rightarrow "$ :
Since the limit tends to zero, we have: For every e>0 there exists an N s.t. diam $\left\{x_{2} m \mid m>=N\right\}<e$. Thus, in particular, $d\left(x_{-} m, x_{-} N\right)<e$ for all $m>=N$. Now use
" $\Leftarrow$ ": part (i).
2.) In the proof of Theorem II. 9 b.), i.e., that $\left(C[a, b], d_{\max }\right)$ is complete, we proceeded as follows: Given a sequence $\left(f_{n}\right)$ of functions $f_{n}:[a, b] \rightarrow \mathbb{R}$, we considered the limit $f(x)=\lim _{n \rightarrow \infty} f_{n}(x)$ for each $x \in[a, b]$ and then showed that $f$ is continuous. Now, let us do an example: Certainly, the functions $f_{n}(x)=x^{n}$ are continuous on $[0,1]$. However, for $f$ we get here (check!)

$$
f(x)= \begin{cases}0 & \text { if } x \in[0,1) \\ 1 & \text { if } x=1\end{cases}
$$

which is clearly not continuous.
So, where have we gone wrong here? Is $\left(C[a, b], d_{\max }\right)$ not complete? Or is the proof wrong (and if so, can we repair it)?
For a solution, click on the following space:
No worries, everything we have done in the lecture is fine. The catch here is that (fn) is not a Cauchy sequence in (C[0,1],dmax). Please check this!
3.) If one would prove Theorem II.10, i.e., that every metric space ( $X, d$ ) has a (up to isometry unique) completion $\left(X^{*}, d^{*}\right)$, then the first few steps are as follows:

- Consider the space $\hat{X}$ of all Cauchy sequences of $X$.
- Define a pseudometric $\rho$ on $\hat{X}$ by $\rho\left(\left(x_{n}\right),\left(y_{n}\right)\right)=\lim _{n \rightarrow \infty} d\left(x_{n}, y_{n}\right)$.
- Then define the following equivalence relation: Two Cauchy sequences $\left(x_{n}\right)$, $\left(y_{n}\right)$ are equivalent, i.e., $\left(x_{n}\right) \sim\left(y_{n}\right)$, iff $\rho\left(\left(x_{n}\right),\left(y_{n}\right)\right)=0$. Use this, as in Exercise sheet 2 Question 2, to obtain a metric space $X^{*}=\hat{X} / \sim$.
- Clearly, $(X, d)$ is an isometric subspace of $X^{*}$. The isometric map sends a point $x \in X$ to the (equivalence class of the) constant sequence ( $x, x, x, x, \ldots$ ).
- The problem is then to show that $X^{*}$ is indeed complete.

In few of these sketchy steps (and although the proof does not apply if one wants to construct $\mathbb{R}$ from $\mathbb{Q}$ ), why does the equation $0.99999 \ldots=1$ hold (or, what does it mean? And is this the reason you were told at school?)?
For a comment, click on the following space:
In the "big" space of all Cauchy sequences, we can interpret 0.99999.... as Cauchy sequence ( $0.9,0.99,0.999,0.9999, \ldots$ ) and 1 as Cauchy sequence ( $1,1,1,1, \ldots$ ). In the above construction, they get identified since the difference between their nth elements is $0.1^{\wedge} \mathrm{n}$ and tends to zero as n goes to infinity. So, on the "completed" space R they belong to the same equivalence class, i.e., they are the same real number: The real number 1 can be represented in two ways using a decimal expansion! I.e., real numbers are not the same as decimal expansions.

