MA30041: Metric Spaces

Self-Assessment Sheet 3: Cauchy Sequences & Completions

1.) (i) Show: A sequence (x_n) in a metric space (X, d) is a Cauchy sequence iff $\forall \varepsilon > 0 \; exists N \in \mathbb{N}$ s.t. $d(x_m, x_N) < \varepsilon$ for all $m \geq N$.
For a solution, click on the the following spaces:

"$\Rightarrow$": ____

"$\Leftarrow$": ____

(ii) Show: A sequence (x_n) in a metric space (X, d) is a Cauchy sequence iff

$$\lim_{N \to \infty} \operatorname{diam}\{x_m \mid m \geq N\} = 0.$$

For a solution, click on the the following spaces:

"$\Rightarrow$": ____

"$\Leftarrow$": ____

2.) In the proof of Theorem II.9 b.), i.e., that $(C[a, b], d_{\max})$ is complete, we proceeded as follows: Given a sequence (f_n) of functions $f_n : [a, b] \to \mathbb{R}$, we considered the limit $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in [a, b]$ and then showed that f is continuous.

Now, let us do an example: Certainly, the functions $f_n(x) = x^n$ are continuous on $[0, 1]$. However, for f we get here (check!)

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1), \\ 1 & \text{if } x = 1, \end{cases}$$

which is clearly not continuous.

So, where have we gone wrong here? Is $(C[a, b], d_{\max})$ not complete? Or is the proof wrong (and if so, can we repair it)?
For a solution, click on the following space:

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Please turn over!

3.) If one would prove Theorem II.10, i.e., that every metric space (X, d) has a (up to isometry unique) completion (X^*, d^*), then the first few steps are as follows:

- Consider the space $\hat{X}$ of all Cauchy sequences of X.
- Define a pseudometric ρ on $\hat{X}$ by $\rho((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n)$.
- Then define the following equivalence relation: Two Cauchy sequences (x_n), (y_n) are equivalent, i.e., $(x_n) \sim (y_n)$, iff $\rho((x_n), (y_n)) = 0$. Use this, as in Exercise sheet 2 Question 2, to obtain a metric space $X^* = \hat{X}/\sim$.
- Clearly, (X, d) is an isometric subspace of X^*. The isometric map sends a point $x \in X$ to the (equivalence class of the) constant sequence $(x, x, x, x, \ldots)$.
- The problem is then to show that X^* is indeed complete.

In few of these sketchy steps (and although the proof does not apply if one wants to construct $\mathbb{R}$ from $\mathbb{Q}$), why does the equation $0.99999\ldots = 1$ hold (or, what does it mean? And is this the reason you were told at school?)?
For a comment, click on the following space:
