## MA30041: Metric Spaces

## Self-Assessment Sheet 2: Further Examples \& Convergent Sequences

1.) A complex number $z=x+i y \in \mathbb{C}$ can be represented as point $(x, y)$ in the plane $\mathbb{R}^{2}$. One can also associate a point $(u, v, w)$ on the unit sphere $\mathbb{S}=\{(u, v, w) \in$ $\left.\mathbb{R}^{3} \mid u^{2}+v^{2}+w^{2}=1\right\}$, with a given point $(x, y)$ in the plane ${ }^{1}$. The associated mapping is called stereographic projection.


We note:

- The equator of the sphere corresponds to the unit circle in the plane.
- The south pole $(0,0,-1)$ corresponds to the origin $(0,0)$.
- A point $(u, v, w)$ on the sphere corresponds to $(x, y)$ if the north pole $(0,0,1)$, $(u, v, w)$ and $(x, y, 0)$ lie on a line.
The stereographic projection, which projects a point $(u, v, w) \in \mathbb{S} \backslash\{0,0,1\}$ to a point of the (complex) plane $z=x+i y \in \mathbb{C} \cong \mathbb{R}^{2}$, and its inverse are given by the following maps (check!):

$$
u=\frac{2 x}{|z|^{2}+1}, \quad v=\frac{2 y}{|z|^{2}+1}, \quad w=\frac{|z|^{2}-1}{|z|^{2}+1}
$$

and

$$
x=\frac{u}{1-w}, \quad y=\frac{v}{1-w} .
$$

Now, think about this construction in terms of metric spaces: What is happening to $\mathbb{C}$ here?
For a solution, click on the the following space:
First note that the unit sphere is a subspace of the three-dimensional space, so the three-dimensional Euclidean metric induces a metric on the sphere, also called the "chordal metric" (why?). Furthermore, the stereographic projection is a bijective map between the plane and the unit sphere without the north pole. So this induces a new metric on C (as on $\mathrm{R}^{2}$ ) which makes the complex plane a bounded metric space!

[^0]2.) Can you find $5(50,500,5000, \ldots)$ metrics on $\mathbb{R}$ ?

For a solution, click on the first of the following spaces:
Proposition 1.4 shows that (i) the sum of two metrics, (ii) a positive multiple of a metric, (iii) the maximum of two metrics, (iv) the minimum of a metric and any positive number and -using Exercise Sheet 2, Question $1--(v)$ any concave function of a metric is again a metric. Click on the next space to see a rather complicated metric on $R$ (the great thing is, we do not have to check the triangle inequality for it!).

Let $d$ be the discrete metric, then surely(!) the following function is a metric on $R:(x, y)$-> $\max \left\{5.221[\mathrm{~d}(\mathrm{x}, \mathrm{y})+\sin (\min \{|x-y|, \mathrm{pi} / 2\})]^{\wedge}(0.7883), \log \left(1+\left[|x-y|^{\wedge}(0.12) /\left(1+|x-y|^{\wedge}(0.12)\right)\right]\right\}\right\}$
3.) If ( $X, d$ ) is a metric space and $Y \subset X$, then $Y$ is made into a metric subspace of $X$ in one way only, namely, by restricting the metric $d$ of $X$ to $Y \times Y$.

Suppose now that $(X, d)$ is a metric space and $Y$ is a proper superset of $X$, i.e., $X \subset Y$ and $X \neq Y$. Is it always possible to define a metric $d^{\prime}$ on $Y$ that is an extension of $d$ (i.e., so that $(X, d)$ is a metric subspace of $\left.\left(Y, d^{\prime}\right)\right)$ ?
For a solution, click on the following space:
(note, we are using the notation $Y-X$ for $Y \backslash X$ here)
There are many ways to do this, e.g.: Let p be any metric on $\mathrm{Y}-\mathrm{X}$ (e.g., the discrete metric). Fix a point $u$ in $X$ and a point $v$ in $Y-X$. Then set $d^{\prime}(x, y)=d(x, y)$ if $x, y$ in $X$, $d^{\prime}(x, y)=p(x, y)$ if $x, y$ in $Y-X$ and $d^{\prime}(x, y)=d(x, u)+1+p(v, y)$ if $x$ in $X$ and $y$ in $Y-X$ (the fourth case, y in X and x in $\mathrm{Y}-\mathrm{X}$ is treated similarly). Then d' is a metric (make a sketch to understand it and check the axioms (MS1)-(MS4)).
4.) We say that a sequence $\left(x_{n}\right) \subset X$ is eventually constant if there exists an $x \in X$ and an $N \in \mathbb{N}$ s.t. $x_{n}=x \forall n \geq N$.
Show: In a discrete metric space $(X, d)$ a sequence $\left(x_{n}\right)$ is convergent iff it is eventually constant.
For a solution, click on the following space:
Since $\mathrm{d}(\mathrm{x}, \mathrm{y})=1$ whenever x and y are not equal, we have that $\mathrm{d}(\mathrm{xn}, \mathrm{x})<0.5$ (or $<0.05$ or $<0.000337$ or $\ldots$ ) holds for all $n=>N$ iff $x n=x$ for all $n=>N$. Thus ( $x n$ ) is eventually constant iff it is convergent.


[^0]:    ${ }^{1}$ We embed the plane into $\mathbb{R}^{3}$ by $(x, y) \mapsto(x, y, 0)$.

