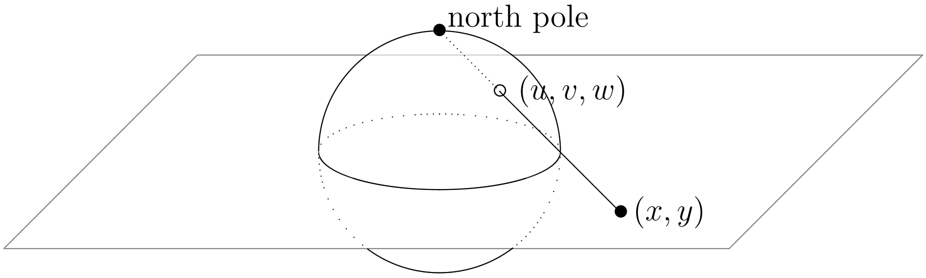
MA30041: Metric Spaces

Self-Assessment Sheet 2: Further Examples & Convergent Sequences

1.) A complex number $z = x + iy \in \mathbb{C}$ can be represented as point (x, y) in the plane $\mathbb{R}^2$. One can also associate a point (u, v, w) on the unit sphere $\mathbb{S} = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\}$, with a given point (x, y) in the plane[1]. The associated mapping is called *stereographic projection.*



We note:

- The equator of the sphere corresponds to the unit circle in the plane.
- The south pole $(0, 0, -1)$ corresponds to the origin $(0, 0)$.
- A point (u, v, w) on the sphere corresponds to (x, y) if the north pole $(0, 0, 1)$, (u, v, w) and $(x, y, 0)$ lie on a line.

The *stereographic projection*, which projects a point $(u, v, w) \in \mathbb{S} \setminus \{0, 0, 1\}$ to a point of the (complex) plane $z = x + iy \in \mathbb{C} \cong \mathbb{R}^2$, and its inverse are given by the following maps (check!):

$$u = \frac{2\,x}{|z|^2 + 1}, \qquad v = \frac{2\,y}{|z|^2 + 1}, \qquad w = \frac{|z|^2 - 1}{|z|^2 + 1}$$

and

$$x = \frac{u}{1 - w}, \qquad y = \frac{v}{1 - w}.$$

Now, think about this construction in terms of metric spaces: What is happening to $\mathbb{C}$ here?

For a solution, click on the the following space:

Please turn over!

[1]We embed the plane into $\mathbb{R}^3$ by $(x, y) \mapsto (x, y, 0)$.

2.) Can you find 5 (50, 500, 5000, ...) metrics on $\mathbb{R}$?
For a solution, click on the first of the following spaces:

3.) If (X, d) is a metric space and $Y \subset X$, then Y is made into a metric subspace of X in one way only, namely, by restricting the metric d of X to $Y \times Y$.

Suppose now that (X, d) is a metric space and Y is a *proper superset* of X, i.e., $X \subset Y$ and $X \neq Y$. Is it always possible to define a metric d' on Y that is an *extension* of d (i.e., so that (X, d) is a metric subspace of (Y, d'))?
For a solution, click on the following space:
(note, we are using the notation $Y - X$ for $Y \setminus X$ here)

4.) We say that a sequence $(x_n) \subset X$ is *eventually constant* if there exists an $x \in X$ and an $N \in \mathbb{N}$ s.t. $x_n = x \; \forall n \geq N$.

Show: In a discrete metric space (X, d) a sequence (x_n) is convergent iff it is eventually constant.
For a solution, click on the following space: