**MA30041: Metric Spaces**

Remarks on the Lectures

# Lecture 29.9.

### Chapter I: Definition & Examples

- Definition *metric space*: [Shirali, Definition 1.2.1 (p. 27f)].
- Theorem I.1: Exercise sheet 1, Question 1.
- *Discrete metric space* with metric $d_{\mathrm{D}}$: [Shirali, Example 1.2.2(v) (p. 30)].
- *Usual* or *standard* metric on $\mathbb{R}$: [Shirali, Example 1.2.2(i) (p. 28)].
- *$n$-dimensional Euclidean metric* $d_{\mathrm{E}}$ (or $d_2$): [Shirali, Example 1.2.2(ii) (p. 28f)].
- Theorem I.2 (*Cauchy-Schwarz inequality*): see [Shirali, Theorem 1.1.4 (p. 25)].
- *Sum* (or *taxicab/Manhattan/New York*) metric $d_1$ on $\mathbb{R}^n$: see [Shirali, Example 1.2.2(iii) (p. 29f)].
- *Maximum* metric $d_\infty$ on $\mathbb{R}^n$: [Shirali, Example 1.2.2(iv) (p. 30)].
- *Barbed wire* or *jungle river* metric on $\mathbb{R}^2$: Self-assessment sheet 1, Question 4.
- *Space of bounded functions* with *uniform* or *supremum* metric: [Shirali, Example 1.2.2(viii) (p. 31f)].

# Lecture 6.10.

### Chapter I: Definition & Examples (cont.)

- Definition *metric subspace*: [Shirali, (p. 28)].
- *Space of continuous functions* with supremum/maximum metric: [Shirali, Example 1.2.2(ix) (p. 32)].
- Definition *product metrics* $d_1$, $_2$ and $d_\infty$: they are called $d$, $d'$ and $d''$ in [Shirali, Definition 6.2.2 & Remark 6.2.3(i) (p. 203f)].
- Definition of *isometry*: [Shirali, Definition 1.5.2 (p. 55)].
- Theorem I.3: *Suppose $Z$ is a set, $(X, d)$ is a metric space and $f : Z \to X$ an injective function. Then, $f$ and $d$ induce a metric on $Z$, namely by $(a, b) \mapsto d(f(a), f(b))$. This metric makes $Z$ an isometric copy of the metric subspace $(f(Z), d)$ of $(X, d)$ and $f$ an isometry.*

- Definition *pseudometric space*: [Shirali, Definition 1.2.5 (p. 36)].
- Examples of pseudometric spaces: see Exercise sheet 1, Questions 2(i) & 4(i).
- Proposition I.4: compare Exercise sheet 2, Question 1 & Self-assessment sheet 2, Question 3.

**Chapter II: Sequences & Completions**

- Definition *convergent/divergent sequence & limit*: [Shirali, Definition 1.3.2 (p. 38)].
- Theorem II.1: [Shirali, Remark 3 (p. 38)].
- Convergence depends on the space $X$ and on the metric $d$: Consider the sequence $\left(\frac{1}{n}\right)$ in $(0,1]$ and $[0,1]$ (equipped with the usual metric) and in $[0,1]$ equipped with the discrete metric.
- Theorem II.2 ("Convergence in product metric space = coordinatewise convergence"): *Let $(X_1, \rho_1)$ and $(X_2, \rho_2)$ be two metric spaces. A sequence $((x_n, y_n)) \subset X_1 \times X_2$ converges to $(x, y)$ iff $x_n \to x$ in $X_1$ and $y_n \to y$ in $X_2$ as $n \to \infty$.*
(implicit in [Shirali, Examples 1.3.3(ii) & (iii) (p. 38f)])
- Definition of *equivalent metrics*: [Shirali, Definition 3.5.4 (p. 120)].
- Definition of *uniformly equivalent metrics*: *Two metrics $d$ and $\tilde{d}$ on $X$ are* uniformly equivalent *if there are constants $C_1, C_2 > 0$ s.t. $\forall x, y \in X$ one has $C_1 \cdot d(x,y) \leq \tilde{d}(x,y) \leq C_2 \cdot d(x,y)$.*
(compare [Shirali, Theorem 3.5.6 (p. 121)])
- Theorem II.3: [Shirali, Example 3.5.7(i) (p. 121) & Remark 6.2.3(ii) (p. 204)].
- Definition of *boundedness & diameter*: [Shirali, Definition 2.1.41 (p. 76)].
- Remark: An unbounded metric space can be made into a bounded one, see [Shirali, Example 3.5.7(ii) (p. 121f)].
- Proposition II.4: see Exercise sheet 2, Question 5.

# Lecture 13.10.

**Chapter II: Sequences & Completions (cont.)**

- Remark: No "monotone sequence theorem" on $\mathbb{C}$.
- Definition *Cauchy sequence*: [Shirali, Definition 1.4.1 (p. 45)].
- Proposition II.5: [Shirali, Proposition 1.4.3 (p. 46)].
- Proposition II.6: *Every Cauchy sequence is bounded.*
- Example: Harmonic series diverges.

- Proposition II.7: [Shirali, Proposition 1.4.7 (p. 48)].
- Remark: $(\frac{1}{n})$ in $(0,1]$ and in $[0,1]$.
- Definition *complete* metric space: [Shirali, Definition 1.4.5 (p. 47)].
- Theorem II.8: see [Shirali, Proposition 1.4.8 & Corollary 1.4.9 (p. 48f)].
- Theorem II.9: [Shirali, Propositions 1.4.12 & 1.4.13 (p. 51f)].
- Theorem II.10 (without proof): [Shirali, Theorem 1.5.3 (p. 55)].

# Lecture 20.10.

**Chapter III: Topology of a Metric Space**

- Definition *open* and *closed balls*: [Shirali, Definition 2.1.1 (p. 64)].
  *Note*: We are using the notation $B_r(x_0)$ and $\overline{B}_r(x_0)$ where [Shirali] uses $S(x_0, r)$ respectively $\overline{S}(x_0, r)$.
- Examples: [Shirali, Examples 2.1.2(i)–(iii) (p. 64f)].
- Definition *neighbourhood*: *Let* $x_0 \in X$ *where* $(X, d)$ *is a metric space. Then* $U(x_0) \subset X$ *is called a* neighbourhood *of* $x_0$ *in* $(X, d)$ *if there exists an* $\varepsilon > 0$ *s.t.* $B_\varepsilon(x_0) \subset U(x_0)$.
  *Note:* This is **not** [Shirali, Definition 2.1.3 (p. 66)].
- Definition *open* set: [Shirali, Definition 2.1.4 (p. 66)].
- Theorem III.1: [Shirali, Theorem 2.1.5 (p. 66)].
- Theorem III.2: [Shirali, Theorem 2.1.7 (p. 67)].
- Remarks: [Shirali, Remark 2.1.8 & Theorem 2.1.9 (p. 67)].
- Side-remark: Topology & every metric space is a topological space (not examinable).
- Definition *interior point* and *interior*: [Shirali, Definition 2.1.12 (p. 69)].
- Remarks: [Shirali, Theorem 2.1.14 (p. 69)].
- Definition *limit point* and *derived set*: [Shirali, Definition 2.1.17 (p. 70)].
- Examples: [Shirali, Examples 2.1.18(i)–(iii) (p. 70)].
- Theorem III.3: [Shirali, Proposition 2.1.19 (p. 70)].
- Corollary III.4: (a) *A finite subset of* $(X, d)$ *has no limit points.*
  (b) [Shirali, Proposition 2.1.20 (p. 71)].
- Definition *closed* set: [Shirali, Definition 2.1.21 (p. 71)].
- Proposition III.5: [Shirali, Proposition 2.1.23 (p. 71)].
- Definition *closure*: [Shirali, Definition 2.1.25 (p. 72)].

# Lecture 27.10.

### Chapter III: Topology of a Metric Space (cont.)

- Proposition III.6: [Shirali, Corollary 2.1.26 (p. 72)].
- Theorem III.7: [Shirali, Theorem 2.1.32 (p. 74)].
- Examples: see [Shirali, Example 2.1.22(iv) (p. 71)].
- Theorem III.8: [Shirali, Corollary 2.1.34 (p. 74)].
- Remark: [Shirali, Remark 2.1.35 (p. 75)].
- Theorem III.9: [Shirali, Theorem 2.12.33 (p. 74)].
- Definition *boundary* (or *frontier*): *Let* $(X, d)$ *be a metric space. Then the* boundary $\partial A$ *(or* $\operatorname{bd} A$*) of a set* $A \subset X$ *is defined as* $\partial A = \operatorname{cl} A \setminus \operatorname{int} A$.
- Examples: Boundary of $[0, 1]$, $\mathbb{Q}$ and clopen sets.
- Proposition III.10: *Let* $(X, d)$ *be a metric space and* $A \subset X$*. Then,* $\operatorname{cl}(A^c) = (\operatorname{int} A)^c$, $\operatorname{int}(A^c) = (\operatorname{cl} A)^c$ *and* $\partial A = \operatorname{cl} A \cap \operatorname{cl}(A^c) = \partial(A^c)$*. In particular,* $\partial A$ *is closed.*
- Theorem III.11: see [Shirali, Proposition 2.1.28 (p. 72)].
- Theorem III.12: [Shirali, Theorem 2.2.2 (p. 80)].

# Lecture 3.11.

### Chapter IV: Continuity

- Definition *continuous* function: [Shirali, Definition 3.1.1 (p. 103f)].
- Theorem IV.1: [Shirali, Theorem 3.1.3 & Propositions 3.1.5 & 3.1.8 (p. 104ff)].
- Theorem IV.2: [Shirali, Theorems 3.1.9 & 3.1.10 (p. 106)].
- Remark (not examinable): *open* and *closed* functions, compare [Shirali, Questions 11 & 12 (p. 146)].
- Theorem IV.3: [Shirali, Theorem 3.1.11 (p. 107)].
- Theorem IV.4: see Exercise sheet 6 Question 2.
- Definition *uniformly continuous* function: [Shirali, Definition 3.4.1 (p. 114)].
- Theorem IV.5: see Exercise sheet 6 Question 3.

# Lecture 10.11.

### Chapter IV: Continuity (cont.)

- Theorem IV.6: see Remark after [Shirali, Definition 3.4.1 (p. 114)] and [Shirali, Theorem 3.4.6 (p. 117)].
- Examples: $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ is continuous but not uniformly continuous; $h : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x + y$ is uniformly continuous; $g : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x \cdot y$ is continuous but not uniformly continuous.
- Theorem IV.7: (Uniform) continuity of $\frac{1}{f}$, $(f + g)$ and $(f \cdot g)$ if $f, g : X \to \mathbb{R}$ are (uniformly) continuous (see M11).
- Theorem IV.8: [Shirali, Theorem 3.4.4 (p. 116)].
- Definition *Lipschitz continuous*: *Let* $(X, d)$ *and* $(Y, \tilde{d})$ *be metric spaces and* $f : X \to Y$*. If there exists a constant* $L > 0$ *s.t.* $\tilde{d}(f(x), f(y)) \leq L \cdot d(x, y)$ $\forall x, y \in X$*, then we say that* $f$ *is* Lipschitz continuous *with* Lipschitz constant $L$*.*
- Proposition IV.9: *A Lipschitz continuous function is uniformly continuous.*
- Definition *homeomorphism*: [Shirali, Definition 3.5.1 (p. 119)].
- Remark: isometry, *uniform isomorphism* and *Lipschitz isomorphism* (not examinable).
- Example: $(-1, 1)$ and $\mathbb{R}$ (both equipped with the usual metric) are homeomorphic (via $f(x) = \tan(\frac{\pi}{2}x)$).
- Proposition IV.10: see [Shirali, Remark 3.5.5 (p. 120)].
- Remark: metrics on $\mathbb{R}^n$, see [Shirali, Example 3.5.7(i) (p. 121)]

# Lecture 17.11.

### Chapter V: Contractions

- Theorem V.1: [Shirali, Propositions 2.2.5 & 2.2.6 (p. 81f)].
- Example: Theorem II.9 revisited.
- Definition *dense* subset: [Shirali, Definition 2.3.12 (p. 84f)].
- Example: [Shirali, Example 2.3.13(i) (p. 85)] and Theorem II.10 revisited.
- Definition *contraction*: [Shirali, Definition 3.7.1 (p. 132f)].
- Definition *fixed point*: [Shirali, Definition 3.7.3 (p. 133)].
- Theorem V.2 (*Banach's Contraction Mapping Principle*, *Banach's Fixed Point Theorem*): [Shirali, Theorem 3.7.4 (p. 133f)].

- Remark: see [Shirali, Remark 3.7.5 (p. 134)].
- Example: *Hausdorff metric* and *Cantor set*, see Question 5 on Exercise sheet 8.
- Definition *solution of a initial value problem*: [Shirali, Definition 3.7.8 (p. 137)].
- Definition *Lipschitz condition*: *We say that $f$ satisfies a* Lipschitz condition *if $\exists M > 0$ s.t. $|f(t,x) - f(t,y)| \leq M\,|x - y|\ \forall (t,x),(t,y) \in R$.*
- Remarks: see [Shirali, Proposition 3.7.11 (p. 138f)] and uniform equivalence of the metrics $d_{\max,\beta}(g_1, g_2) = \max_{t\in I} |g_1(t) - g_2(t)|\, \mathrm{e}^{-\beta t}$ and $d_{\max}$.
- Theorem V.3 (*Picard's Theorem*): compare [Shirali, Theorem 3.7.12 (p. 139f)].

# Lecture 24.11.

### Chapter VI: Compactness

*Note: Unfortunately,* [Shirali] *does not distinguish between compactness and sequential compactness!*

- Definition *compact* metric space and *compact* set: [Shirali, Definition 5.1.1 (p. 171)].
- Examples: $\mathbb{R}$, $(-1, 1)$ are not compact.
- Definition *(finite) $r$-net*: [Shirali, Definition 5.1.7 (p. 173)].
- Definition *total boundedness*: [Shirali, Definition 5.1.8 (p. 174)].
- Proposition VI.1: [Shirali, Proposition 5.1.10 (p. 175)].
- Theorem VI.2: [Shirali, Theorem 5.1.12 (p. 175f)].
- Corollary VI.3: *Let $(X, d)$, $(Y, \tilde{d})$ be metric spaces. If $f : X \to Y$ is a surjective uniformly continuous function and $X$ is totally bounded, then $Y$ is totally bounded (and hence also bounded).*
- Definition *sequentially compact*: *Suppose that $(X, d)$ is a metric space. A subset $K \subset X$ is said to be* sequentially compact *if any sequence $(x_n) \subset K$ has a subsequence which converges to a point in $X$.*
- Theorem VI.4: see [Shirali, Theorem 5.1.17 (i.e., Propositions 5.1.13 & 5.1.14, Theorem 5.1.16) (p. 176ff)].
- Proposition VI.5: [Shirali, Corollaries 5.2.4 & 5.2.5 (p. 180)].
- Proposition VI.6 (Heine-Borel Theorem): see [Shirali, Remark 5.1.11 & Section 5.5(d) (p. 175 & 194)]
- Proposition VI.7: *Let $A_1, A_2$ be nonempty subsets of a sequentially compact metric space $(X, d)$. Then $\exists x \in \operatorname{cl} A_1$ and $y \in \operatorname{cl} A_2$ s.t.* $\operatorname{dist}(A_1, A_2) = d(x, y)$.
- Theorem VI.8: [Shirali, Theorem 5.3.10 (p. 184)].

# Lecture 1.12.

### Chapter VI: Compactness

- Theorem VI.9: *Suppose that* $f : (X, d) \to (Y, \tilde{d})$ *is continuous and* $K \subset X$ *is sequentially compact. Then* $f(K)$ *is sequentially compact.*
- Theorem VI.10: [Shirali, Theorem 5.3.8 (p. 183f)].
- Theorem VI.11: see [Shirali, Theorems 5.2.2 & 5.2.3 (p. 179f)].

### Chapter VII: Connectedness

- Definition *disconnected* & *connected*: see [Shirali, Definition 4.1.1 & Theorem 4.1.3 (p. 156f)]
  *Note:* We are using the equivalent condition [Shirali, Theorem 4.1.3(ii)] as definition for being *disconnected*.
- Proposition VII.1 & Corollary VII.2: [Shirali, Theorem 4.1.3 (p. 157f)].
- Proposition VII.3: [Shirali, Theorem 4.1.6 & Corollary 4.1.7 (p. 159)].
- Examples: [Shirali, Examples 4.1.2(ii) & (iii) (p. 157)].
- Theorem VII.4: [Shirali, Theorem 4.1.13 (p. 161)].
- Theorem VII.5: [Shirali, Theorem 4.1.4 (p. 158)].
- Theorem VII.6: [Shirali, Theorem 4.1.8 (p. 159f)].
- Definition *path connected*: [Shirali, Definition 4.3.1 (p. 165)].
- Theorem VII.7: [Shirali, Theorem 4.3.4 (p. 166)].
- Example: [Shirali, Example 4.3.2(i) (p. 165)].

# References

[Shirali] S. Shirali & H.L. Vasudeva : Metric Spaces; Springer (2006); library: ebook