MA30041: Metric Spaces

Proof of Theorem VII.5

Theorem VII.5. *A subset $I \subset \mathbb{R}$ is connected iff I is an interval*[1]*.*

Proof: "$\Rightarrow$": If $I = \varnothing = (x, x)$ or $I = \{x\} = [x, x]$, then there is nothing to prove. So we can assume that I contains at least two points.

We use contraposition, so assume that I is not an interval. Set $a = \inf I$ and $b = \sup I$, and note that $-\infty \leq a < b \leq \infty$. Suppose for simplicity (why can we do that?) that $a, b \notin I$, i.e., $I \subset (a, b)$. Since $I \neq (a, b)$, there is an $x \in (a, b)$ s.t. $x \notin I$. By the definition of infimum and supremeum $I \cap (a, x) \neq \varnothing$ and $I \cap (x, b) \neq \varnothing$. Also note that $I \subset (a, x) \cup (x, b)$ (two nonempty disjoint open sets!). Hence, I is disconnected.

"$\Leftarrow$": Seeking a contradiction, let I be an interval which is disconnected. Then there are nonempty disjoint open sets $U, V \subset \mathbb{R}$ s.t. $U \cap I \neq \varnothing$,$V \cap I \neq \varnothing$ and $I \subset U \cup V$. So, $G_1 = I \cap U$ and $G_2 = I \cap V$ are open in I, nonempty disjoint sets s.t. $I = G_1 \cup G_2$. Let $x_1 \in G_1$ and $x_2 \in G_2$. W.l.o.g. we may assume that $x_1 < x_2$. Since I is an interval, it follows that $(x_1, x_2) \subset I$. Let $A \subset I$ be defined by

$$A = \{z \in I \mid (x_1, z) \subset G_1\}.$$

Since G_1 is open in I, it follows that $A \neq \varnothing$. Since G_2 is open in I, we conclude that $x_2 \notin A$. So, A is bounded above by some $c < x_2$. Thus, $y = \sup A$ is a finite number that belongs to $(x_1, c] \subset I$. Two cases are possible: either $y \in G_1$ or $y \in G_2$:

- Suppose $y \in G_1$. Since $y > x_1$ and G_1 is open, there is a $\delta > 0$ s.t. $y - \delta > x_1$ and $(y - \delta, y + \delta) \subset G_1$. Since $y = \sup A$, we have $\left(x_1, y - \frac{\delta}{2}\right) \subset G_1$ and thus $(x_1, y + \delta) = \left(x_1, y - \frac{\delta}{2}\right) \cup (y - \delta, y + \delta) \subset G_1$. But then $y \neq \sup A$. Thus, this case is not possible.
- Suppose $y \in G_2$. Then, since G_2 is open, there is a $\delta > 0$ s.t. $(y - \delta, y + \delta) \subset G_2$. But then $\left(y - \delta, y - \frac{\delta}{2}\right) \subset G_1$, hence $G_1 \cap G_2 \neq \varnothing$, a contradiction. □

[1] $I \subset \mathbb{R}$ is called an *interval* if when $x, y \in I$ with $x < y$, then $z \in I$ for all $x < z < y$, i.e., $(x, y) \subset I$.