MA30041: Metric Spaces

FAQs (update: January 10, 2009)

- *In my notes I have no proof for Corollary III.4 and was wondering whether we actually did one because I have checked all worksheets and next pages of lecture notes and I can't find one. If we don't prove it, we don't need the proof for the exam then?*

  The proof is (implicitly) contained in the previous proof, the proof of Theorem III.3. There we actually construct such a sequence of distinct points converging to a limit point (the converse, namely, that the limit of such a converging sequence of distinct points is a limit point, is clear).

- *I have just 2 small things to ask you:* 6.1.

  1. *If your metric space was* $[0, 1]$ *then as whole space is both open and closed, but how is it open? Also same with* $(0, 1)$ *but closed?*
  2. *Sheet 3 Qu 5i) why does* $f_m(x_n) < \frac{1}{4}$ *mean* $|f_n(x_n) - f_m(x_n)| > 1/2$ *for all* $m \geq 5n$*?*

  1. That is that "subspace-topology"-game, see Theorem III.12 (for some further examples, also see Self-assessment Sheet 5): A set A in the subspace Y is open/closed if you can find an open/closed set A' in the original space X s.t. $A = A' \cap Y$. And you can also say that the whole space is by definition clopen: open, since any ball (around some point) in that space, is contained in the space, and closed since any limit of a *convergent sequence in that space* (so we know there is a limit in that space) belongs, trivially, to that space.
  2. Because by our choice of f_n and x_n we have $f_n(x_n) = \frac{3}{4}$ for all n.

- *I have some questions on the lecture notes this time :)* 31.12.

  1. *When defining a sequence in the rationals that converges to an irrational, is it right that you can define it such that as n increases you can just add more decimal places of that irrational number (so as it is finite can be written as a fraction) like your example* $\sqrt{2}$ *(1, 1.4, 1.41, 1.414, 1.4142, ...).*
  2. *If showing that the limit points of the rationals is the reals we need to show there is a rational in any epsilon ball around any point in the reals. How do we show this precisely as we know it works?*
  3. *I am not sure I fully understand the alternative definition of continuous with the delta and epsilon balls, how is it that it is equivalent to the other definition of continuous at point x.*
  4. *In the first example of the continuity chapter you have* $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \cos(x)$*. I can see why* $f^{-1}([0, 1]) = \bigcup_{k \in \mathbb{Z}} [-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k]$ *but you then said this was equal to* $f^{-1}([0, \infty))$ *but* $(1, \infty)$ *not mapped to by* f *so wouldn't be in the inverse image?*

5. *I don't understand the proof of ii) implies iii) of Theorem IV.1, particularly the fact that as "x_n in $f^{-1}(U)$ implies $f(x_n)$ in U for all $n \geq N$" for all possible nbhds of x means $f(x_n)$ converges to $f(x)$.*
6. *Can we assume or do we need to know how to prove: $f^{-1}(F) = X \setminus f^{-1}(F^c)$? and is it true that $f(F) = X \setminus f(F^c)$?*
7. *You did an example of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \exp(x) \cdot \cos(x)$ cts but $f((-\infty, 0)) = [\frac{-1}{\sqrt{2}} \cdot \exp(\frac{3\pi}{4}), 1)$, would we be expected to know how to find such a limit because I would probably have no clue how to do it?*
8. *In one remark you said the map $\mathrm{id} : (\mathbb{R}, d_D) \to (\mathbb{R}, d_E)$ is unif cts but not Lipschitz cts, how would u prove this (how do you show there is no L st $d_E(x, y) \leq L \cdot d_D(x, y)$)?*

1. In plain words, convergence on $\mathbb{R}$ is "just adding more decimal places", compare the definition ("(x_n) converges to x if for every $\varepsilon > 0$ there exists an N s.t. $|x - x_n| < \varepsilon$ for all $n \geq N$"). Well, to be a bit more precise, you have to take things like $0.9999\ldots = 1.0000\ldots$ (at least for rationals!) into account. Also note (but you surely are aware of that) that this does *not* mean that number of correct decimal places increases if you go from x_n to x_{n+1}, the number of correct decimal places in (x_n) just increases somehow.
2. An example (actually your example from part 1.): Say, we want to find that a rational in the ball of radius $\varepsilon = 0.001$ around $\sqrt{2}$, then clearly $1.4142 = 14142/1000$ is such a rational. If you take a ball of radius $\varepsilon = 10^{-5}$, $\varepsilon = 10^{-8}$, $\varepsilon = 10^{-N}$ or a general ε, what rational number can you pick?
3. Simply reformulate the usual definition in term of balls: $f : (X, d) \to (Y, d')$ is continuous at x, if for every $\varepsilon > 0$ there exists a $\delta > 0$ s.t. $d(x, y) < \delta$ (i.e., $y \in B_\delta^{(X,d)}(x)$) implies $d'(f(x), f(y)) < \varepsilon$ (i.e., $f(y) \in B_\varepsilon^{(Y,d')}(f(x))$) – or, to make it short, if $f\left(B_\delta^{(X,d)}(x)\right) \subset B_\varepsilon^{(Y,d')}(f(x))$.

4.–6. First recall the definition of the inverse image: If $f : X \to Y$ and $A \subset Y$, then $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. From this, it is clear that we also have

$$f^{-1}(A_1 \cup A_2) = \{x \in X \mid f(x) \in A_1 \cup A_2\} = \\ \{x \in X \mid f(x) \in A_1\} \cup \{x \in X \mid f(x) \in A_2\} = f^{-1}(A_1) \cup f^{-1}(A_2).$$

4. In this example, we have $f^{-1}((1, \infty)) = \{x \in \mathbb{R} \mid \cos(x) > 1\} = \varnothing$ (since cosine is bounded between -1 and 1), $f^{-1}([0, \infty)) = \{x \in \mathbb{R} \mid \cos(x) \geq 0\} = \{x \in \mathbb{R} \mid 0 \leq \cos(x) \leq 1\} = f^{-1}([0, 1])$.
5. That "x_n in $f^{-1}(U)$ implies $f(x_n)$ in U" follows from the definition of the inverse image. A neighbourhood U (of a point x) means that we can find, for some $\varepsilon = \varepsilon(U, x) > 0$, an epsilon ball $B_\varepsilon(x)$ contained in U; thus instead of "for all possible neighbourhoods of $x\ldots$" we can also say: "for every $\varepsilon > 0$" there exists an N (instead of "for every neighbourhood U of x there exists an N") s.t. $d'(f(x_n), f(x)) < \varepsilon$ for all $n \geq N$ – which is the definition of convergence.
6. You can assume AND you need to know how to prove $f^{-1}(F) = X \setminus f^{-1}(F^c)$ (see the remark above)!!

For the second part take $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ and you should immediately see that $f(F) \neq X \setminus f(F^c)$ (for any set $F \subset \mathbb{R}$!!).

7. You do not know how to find a (global) maximum/minimum of a continuous function on $\mathbb{R}$?? You certainly already knew that in your schooldays(!) after learning how to differentiate functions!
8. The discrete metric $d_D(x, y)$ is bounded by 1 (i.e., $d_D(x, y) \leq 1$ for all x, y), while $d_E(0, n) = n$ for all $n \in \mathbb{N}$. So, can you find such a constant L?

- *I think this is the last of my main questions yay!* 31.12.

10. *Why is* $[-1, 1]$ *not homeomorphic to* $\mathbb{R}$*?*
11. *When showing* $C[a, b]$ *complete using Theorem 5.1 why is* $C[a, b]$ *closed?*
12. *Let* $f : X \to X$ *s.t.* $d(f(x), f(y)) < d(x, y)$ *for all* x *not equal to* y*. Why is it that it only has one or no fixed points? I can see how if f was contraction there would be one but not sure about other cases?*
13. *On showing* $T(\Psi)(t) = x_0 + \int_{t_0}^{t} f(s, \Psi(s))\,\mathrm{d}s$ *is a contraction: are we assuming* f *has a Lipschitz condition? Why is it that* $t \geq t_0 - \log(2)/\beta$*?* T *is only a contraction w.r.t.* $d_{max,\beta}$*?*
14. *In the Statement of Picard's theorem what are the intervals? Just not sure if I have them down right?*
15. *Proposition 6.1 – does* A *finite imply that* $\operatorname{diam}(A)$ *is finite?*
16. *Why is it that* $\mathbb{R}$ *is not totally bounded w.r.t.* d *and* $d' = d/(1+d)$ *but is bounded w.r.t.* d' *and not w.r.t.* d*?*
17. *Theorem 6.4 – is it true that if a Cauchy seq has a Cauchy subseq that converges that the initial Cauchy sequence must converge to the same limit?*
18. *Proposition 6.5 i): do we need* X *to be complete?*
19. *Why is a discrete metric space seq compact iff finite?*
20. *Can a sequentially compact set be considered as a metric space in itself?*
21. *Proposition 6.6 – Last two lines, I don't understand how the* r*-net works?*
22. *Is it true that any subset of a totally bounded set is totally bounded?*
23. *Theorem 6.11 – Why is there only one point in the infinite intersection? Is it because* $\operatorname{diam} = 0$*?*
24. *Proposition 7.1 – Why is it that the only clopen sets in* X *are empty and* X *itself?*
25. *Theorem 7.4 –* $U \subset V \subset \operatorname{cl} U$, U *dense in* $\operatorname{cl} U$*: does this imply due to* $V \subset \operatorname{cl} U$ *that* U *is dense in* V*?*
26. *Why is* $V = \{(0, 0)\} \cup \{(x, \sin(1/x)) \mid x \in (0, 1]\}$ *connected but not path connected?*
27. *Theorem 7.5 – Why exclude* a, b *at the beginning? Is it as they can be infinite?*

What happened to Question 9.?

10. Using Theorem VI.9, one can deduce that there is no continuous surjective map $[-1,1] \to \mathbb{R}$.
11. We actually show that $C[a,b]$ is closed in the proof of Theorem II.9: Using the sequential characterisation of closedness, we show that the limit (which – by the first part of Theorem II.9 – belongs to the complete metric space $B[a,b]$ of bounded functions) of a converging sequence of continuous functions is also continuous. Thus, $C[a,b]$ is a closed subset of the complete metric space $B[a,b]$!
12. Suppose it has fixed points $x = f(x)$ and $y = f(y)$. Then (since they are fixed points) you have $d(f(x), f(y)) = d(x,y)$, however, they must be equal $x = y$, otherwise one would have $d(f(x), f(y)) < d(x,y)$. Thus, f cannot have more than one fixed point. That the case of no fixed point occurs is done by an example ($f(x) = x + 1/x$).
13. Yes, we are assuming the Lipschitz condition. For the rest of the question, see the answers to questions on Picard's Theorem in the FAQ-file `faq.pdf` on the course website/Moodle.
14. Compare the remarks on the model solution of Question 4 Exercise sheet 8 (p. 6f).
15. Yes. In that case one has $\operatorname{diam} A = \sup\{d(x,y) | x, y \in A\} = \max\{d(x,y) | x, y \in A\}$ (the second equality arises because A is finite), and clearly $\operatorname{diam} A$ is finite since $d(x,y)$ is finite for any pair x, y.
16. Not totally bounded: Let $\{y_1, \dots, y_n\}$ be a finite set. Then, the number $z = 2 + \max\{y_1, \dots, y_n\}$ is *not* in the union $\bigcup_{j=1}^{n} B_{\frac{1}{2}}(y_j)$ (balls w.r.t. either d or d'); consequently, there is no finite $\frac{1}{2}$-net (w.r.t. either d or d') of $\mathbb{R}$.
    Boundedness: Clearly, the diameter of $\mathbb{R}$ w.r.t. d' is 1 (noting $d'(x,y) < 1$ for all x, y and $d'(0,n) \to 1$ for $n \to \infty$), while $\mathbb{R}$ is unbounded w.r.t. d (since $d(0,n) = n$ for all $n \in \mathbb{N}$).
17. Yes. You can show this using the definitions of Cauchy sequence, converging sequence and a standard $\frac{\varepsilon}{2}$-argument.
18. X (sequentially) compact is equivalent to X complete and totally bounded (by Theorem VI.4).
19. Total boundedness!
20. Yes. In fact, any subset of a metric space can be regard as metric space (i.e., subspace).
21. The point is that for any bounded set in $\mathbb{R}^n$ you can find a finite r-net, and this r-net is what you would do anyway: You enclose the set in question into a "box" (i.e., a rectangle/cuboid/hypercuboid or square/cube/hypercube) and then divide this "box" up into small "boxes" of small enough sidelength (so that each "small box" fits into a ball of radius r). Then, count the "boxes" and see that there are finitely many for any r.
22. Yes, although there is a little "trick" in the proof: Let Y be a nonempty subset of a totally bounded metric space/set X and let $\{x_1, x_2, \dots, x_n\}$ be a finite $\frac{1}{2}r$-net in/of X. Since Y is nonempty, $Y \cap B_{\frac{1}{2}r}(x_j)$ is nonempty for

at least one j, (where $1 \le j \le n$). Choose $y_j \in Y \cap B_{\frac{1}{2}r}(x_j)$ in each such nonempty intersection. If $y \in Y$, then $y \in B_{\frac{1}{2}r}(x_j)$ for some j, and hence $d(y, y_j) \le d(y, x_j) + d(x_j, y_j) < r$, i.e., we have found a finite r-net for Y. Since this works for any r, Y is totally bounded.

23. Yes, see the model solution of Question 1(iv) on Exercise sheet 9.
24. The statement is: The only clopen subsets of a connected(!) metric space X are the empty set and X itself. If there is proper clopen subset of X, then (using the definition) X is disconnected.
25. Yes.
26. Connected: Use Theorem VII.4 and see Question 4(ii) on Exercise sheet 5.
Not path connected: recall that the path has to be a continuous function, in this case a continuous function $\gamma : [0,1] \to \mathbb{R}^2$. Now show that there is no continuous function with $\gamma(0) = (0,0)$ and $\gamma(1) = (s, \sin\frac{1}{s})$ for some $s > 0$: Let $r = \sup\{t \in [0,1] \mid \gamma(t) = (0,0)\}$ (i.e., the "path" γ changes from the singleton $(0,0)$ to the graph $U = \{(x, \sin(1/x)) \mid x \in (0,1]\}$ at r "for the last time"). Note that the set $\{t \in [0,1] \mid \gamma(t) = (0,0)\}$ is closed (see Question 1 on Exercise sheet 8), thus $r < 1$ and $\gamma(r) = (0,0)$. Now, consider for some $\delta > 0$, the graph of $\gamma(t)$ for $t \in [r, r+\delta)$. Since $\gamma(t) \ne (0,0)$ for $t > r$, we have $\gamma(r + \frac{\delta}{2}) = (1/a, \sin(a))$ for some $a > 0$. It is then clear that γ can only be continuous if all of $U' = \{(x, \sin(1/x)) \mid x \in [1/(a+2\pi), 1/a]\}$ is contained in the graph $\{\gamma(t) \mid t \in [r, r+\delta/2]\}$: Note that if γ is continuous, then the function that maps t to the first coordinate of $\gamma(t)$ is also continuous and observe that the continuous image of a closed and bounded interval (i.e., compact and connected subset of $\mathbb{R}$) is a closed and bounded interval (here, the image of $[r, r+\delta/2]$ must contain $[0, 1/a]$). But U' contains "a period of sine", i.e., the second coordinate contains all values between -1 and 1. So, there is a $t_1 \in [r, r+\delta)$ s.t. $d(\gamma(t_1), (0,0)) \ge \frac{1}{2}$. Since this holds for any $\delta > 0$, the function γ is not continuous and thus no path, i.e., V is not path connected.
27. It is really simplicity; if you want to be precise, you have to consider (but all work with the same argument, check!) all cases (a,b), $[a,b)$, $(a,b]$, $[a,b]$, $(-\infty,b)$, $(-\infty,b]$, (a,∞), $[a,\infty)$ and $(-\infty,\infty)$.

- *I have several questions on chapters 2, 3 and 5 (I haven't got round to counting chapters 4, 6 and 7 since I didn't want you to be swamped by too many all at once). I hope you can help me please. I have subdivided them to make it a little easier for you:* 6.1.
  - *CHAPTER 2*
    1. *the convergent definition – is this uniform convergence because my N only depends on ε and not x as well?*
    2. *product space $X_1 + X_2 + \ldots + X_n$ – is it the space of all metric spaces (X_1, d_1), ..., (X_n, d_n)?*
    (I am thinking about this as if $X_1 = C[a,b]$ and $X_2 = C[b,d]$ then $X_1 + X_2 = C[a,d]$ with metrics d_1 and d_2 applicable)

3. *proposition 3.6 – why is* $d(x_n, x_N) + d(x_N, x_m) \leq M + 1$ *since for defining* M *I thought that it was for all* $n \leq N$ and $m \leq N$ *so then it would be* $d(x_n, x_N) + d(x_N, x_m) \leq 2M$*???*
4. d *and* ρ *being equivalent – does this mean that if* $1/n \to 0$ *wrt* d *then* $1/n \to 0$ *wrt* ρ*?? So they must have same Cauchy sequences and the same limit for that particular sequence, we can't have* $1/n \to 0$ *wrt* d *and* $1/n \to a$ *wrt* p*?*
5. *If we complete* $(0, 1]$ *then do we say that* $X^* = \{0\} \cup (0, 1]$ *and keep the same metric function or do we define new metric function? Similary with the rationals,* $\mathbb{Q}$*, do we have* $X^* = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ *with same metric?*

– *CHAPTER 3*

1. *How does showing* $x_k \in U \setminus \{x\}$ *for all* k *prove that* $A \cap U$ *is infinte for all* U*?*
2. *Theorem III.12(ii): is* F *contained in* X*? I have in my notes* F *is contained in* Y *so am slightly confused, I think I have copied it down wrong!*
3. *How come if balls can't be contained at just one point then how come the open ball wrt the discrete metric* d_D *is:* $B_r(x) = \{x\}$ *if* $r < 1$ *and* X *if* $r \geq 1$*???*

– *CHAPTER 5*

1. *Why in Picard's theorem is* $I = [t_0 - \frac{\ln 2}{2M}, t_0 + a]$ *and not* $[t_0 - a, t_0 + a]$*?*
2. *Why is* $t \geq t_0 - \frac{\ln 2}{\beta}$ *in the contraction proof we did and not* $t \geq t_0 + \frac{\ln 2}{\beta}$*? (I can't get it to work!)*

Thanks for subdividing, now let's see:

– CHAPTER 2

1. We are talking about sequences (of points) here, not about sequences of functions (functional analysis) where pointwise and uniform convergence are important concepts!
2. NO! The "model" product space is $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$, i.e., one takes the product of n copies of the reals with the standard metric, $(\mathbb{R}, d)$. Now, what are the points/elements of a product space $X_1 \times X_2 \times \ldots \times X_n$: These are all n-tuples $(a_1, ..., a_n)$ where $a_i \in X_i$ for all i (well, in the vector space $\mathbb{R}^n$ this is a bit easier since each factor X_i is just $\mathbb{R}$ – however, in general that is not the case and we also do not necessarily have a vector space). And the metric is given by one of the product metrics, the max-, sum- or Euclidean metric (we have shown that "for all practical questions we are interested in", e.g., convergence, completeness, topology (open & closed) etc., they give the same result). So, if we speak of a product (metric) space then we also mean this "product space" $X_1 \times X_2 \times \ldots \times X_n = \{(a_1, ...a_n) \,|\, a_i \in X_i\}$ together with one of these product metrics!
3. Are you talking about Proposition II.6 (not III.6)? If so, you are right – in the case $n, m \leq N$, and then you have to consider the other cases $n \leq N < m$, $m \leq N < n$ and $N < n, m$. However, I hope that I have stated my proof correctly: Take N s.t. $d(x_n, x_m) < 1$ for all $n, m \geq N$

and set $M = \text{diam}\{x_1, ..., x_N\}$. Then, in the case $n, m \leq N$ we have $d(x_n, x_m) \leq M$, in the case $n, m \leq N$ we have $d(x_n, x_m) < 1$ and the remaining case ($n \leq N \leq m$ or $m \leq N \leq n$) is the calculation in your question!

4. Yes, equivalent means that ρ and d have the same convergent sequences (with the same limit). However, if the spaces are not complete, they do not necessarily have the same Cauchy sequences, see Question 4 on Exercise sheet 3.
5. Well, that depends what you mean by "the same" If you are precise then it is not the same since already the domains of the metric function are different! However, the Completion Theorem (Theorem II.10) shows that metric function on the ("smaller") incomplete space "fixes" (up to isometry) the metric function on the ("bigger") complete space. Thus, it makes sense to say that they are "the same" (and that's what one usually does).

– CHAPTER 3

1. Well, we also show that all x_k are in A *and* that they are pairwise different! So, we have infinitely many pairwise different points (namely, the x_ks), that belong to A and to $U \setminus \{x\}$.
2. It should read: "A is closed in Y iff there exists a closed set F in X such that $A = F \cap Y$."
3. Your questions sounds like the first part is some statement (something I have said?) about balls in an Euclidean space $\mathbb{R}^n$. Clearly, your open ball w.r.t. the discrete metric is okay (just write down the definition of a ball and use the discrete metric). So, I am not sure what to answer here, can you give some more information (especially what you have in mind in the first part).

– CHAPTER 5

1. Yes, you can also do that (it is even better!), see the remarks to the model solution of Question 4 on Exercise sheet 8 (p. 6/7 there).
2. See the FAQs `faq.pdf` on the course website/Moodle site, you can find the answer there (one of the Picard's theorem question).

- *(Continuation of the previous question)* 7.1.

– *CHAPTER 2*

3. *Yes, I was talking about propostion 2.6, sorry I spelt it wrong! So are we saying that with our M defined,*

$$\begin{aligned} d(x_n, x_m) &< 1 \quad \text{if } n, m \leq N, \\ d(x_n, x_m) &< M + 1 \quad \text{if } n \leq N \leq m \text{ or } m \leq N \leq n, \\ d(x_n, x_m) &< 2M \quad \text{if } n, m \leq N? \end{aligned}$$

So then would it be ok to choose $K := \min\{1, M+1, 2M\}$ and say $d(x_n, x_m) < K$ for all $n, m \geq K$??

5. *OK so are you saying we use the same metric function but adapt it so it is applicable on X^* then?*

- *CHAPTER 3*

  3. *Yes you said in a lecture in this chapter about balls can't be contained at one point-the balls are neither open nor closed. I wrote it down, but you didn't – I might have misheard! But you explained that balls have to be greater than* 1 *point because something about if* $B_r(x) = \{x\}$ *and if* x *is a limit point of* A *then* $B_r(x) \setminus \{x\} \cap A$ *is nonempty. But for the open ball contained at just one point is empty!*
     I didn't follow how if it was true that balls cannot be contained at one point then how come if using the discrete metric that if $r < 1$ *that* $B_r(x) = \{x\}$ *is allowed to be a ball?*

- CHAPTER 2

  3. It would be o.k. to choose $K = \max\{1, M + 1, 2M\}$ (not min!) and say $d(x_n, x_m) \leq K$ for *all* n, m (it should read $d(x_n, x_m) \leq 2M$ if $n, m \leq N$). But observe that we even have $d(x_n, x_m) \leq M$ if $n, m \leq N$ and thus $K = \max\{1, M + 1, M\} = M + 1$ — and $d(x_n, x_m) < M + 1$ for all n, m follows.
  5. We extend it to a "bigger" space and the "surprising" thing is that there is essentially only one such extension possible (compare that with the general case as in Self-assessment sheet 2 Question 3 on the course website/Moodle site: one would expect that there are many extensions of a metric to a superset – the catch here is that the original space is dense in its completion thus there is essentially only one extension here!).

- CHAPTER 3

  3. Ahh, I think I made a remark about the values of r and whether or why we use (the convention) $r > 0$ and not $r \geq 0$ in the definition of balls: If one allows $r = 0$, then the open ball of radius 0 is always the (pathological) empty set and even the centre of such an open ball does not belong to the ball, while the closed ball of radius 0 around x is the singleton set $\{x\}$ – both statements would be fine, the empty set is open and the singletons are always closed, the only odd thing is that the centre of an (open) ball would not belong to the ball. Thus, we only allow values $r > 0$, where this oddity does not happen. But it really is a convention whether one includes the value $r = 0$ or not (but one then, if one wants to be precise, might need to exclude the case $r = 0$ in later definitions)!

- *I have more questions, this time from chapters 4, 6 and 7 with a few on the self-assessment sheets.* **9.1.**

  - *CHAPTER 4*

    1. *In theorem 4.1 part iv) is* A *contained in* X*.*
       Also in the proof of iv)⇒i), does $B_\delta(x) \cap A = \varnothing$ *imply that* $B_\delta(x) \cap A^c \neq \varnothing$ *which implies that* $f(B_\delta(x)) \cap (f(A))^c \neq \varnothing$*?*
    2. *In theorem 4.2, in ii)⇒iii) then does* $f^{-1}(F^c)$ *open implies* $f^{-1}(X \setminus F)$ *is open* $\Rightarrow f^{-1}(F) = f^{-1}(Y \setminus F^c) = f^{-1}(Y) \setminus f^{-1}(F^c) = X \setminus f^{-1}(F^c)$ *is open??*

In i)⇒ii): If G is open in Y, shouldn't G be a neighbourhood of any point in Y not G? I'm confused!

3. *In theorem 4.3 i)⇒ii), how do we know* $\mathrm{cl}\,U$ *is closed subset of Y – I thought* $\mathrm{cl}\,U$ *was only a closed subset of Y if* $\mathrm{cl}\,U = U$*?*
4. *In theorem 4.4, is A and U closed or open as we want?*
5. *Theorem 4.5 – how can showing f uniformly continuous* $\Rightarrow \mathrm{dist}(f(U), f(V)) > 0 \Rightarrow \mathrm{dist}(U, V) > 0$ *show ii) if we want* $\mathrm{dist}(f(U), f(V)) = 0 \Rightarrow \mathrm{dist}(U, V) = 0$ *???*
   Why if $x \in f^{-1}(f(U))$ and $y \in f^{-1}(f(V))$ then $d(x, y) \geq \delta$?
   Basically, for 4.5 I understand the proof of the claim and the claim itself BUT I just don't get i)⇒ii) or ii)⇒iii)!!!
6. *Example in lectures, $X = [0, 2] = Y$ and $f(x)$ is 0 if $0 \leq x \leq 1$ and $x - 1$ if $1 < x \leq 2$ but $f(x)$ is in $[0, 1]$!!! Why is we said $Y = [0, 2]$!!!*
7. *Another example in lectures, $f(x) = exp(x)cos(x)$ with $f : \mathbb{R} \to \mathbb{R}$. How is it that $f((-infinity, 0)) = [-(1/\sqrt{2})\exp(3\pi/4), 1)$? Or are we just saying that it is but not needing to know why for this since it is complicated example?*

- *CHAPTER 6*
  1. *For the second definition of an r-net. Isn't it true that for all A in X, since* $\mathrm{dist}(x, A) = 0$ *for at least one $x \in X$? So A is dense in X?*
  2. *For totally bounded definition are we saying that if there exists only one finite r-net for X then it is totally bounded (even if this is the only one we can find)???*
  3. $[0, 1]$ *in $\mathbb{R}$ sequentially compact then a finite subcovering can be found – but I can't find one! I'm lost!*
- *CHAPTER 7*
  1. *In previous exam papers, they have said that the definition of disconnected was your alternative phrasing of X contained in $G_1 \cup G_2$ and $X \cap G_1$ is nonempty and $X \cap G_2$is nonempty but they have $X \cap G_1 \cap G_2$ as empty – is the latter equivalent to saying $G_1 \cap G_2$ is empty???*
  2. *How do we represent connectedness in terms of intersections and unions? I'm confused how that works, is it that X isn't contained in the union of the open sets or is $G_1 \cap G_2$ nonempty????*
  3. *In 7.3, is both the proofs by contradiction?*
  4. *7.8 - Aaaaargh! Not following AT ALL! Plus it wasn't in lecture notes and only on problem sheets so is this proof important to know for exam?*
  5. *The example of connected but not path connected in lecture notes, of $V = \{(0, 0)\} \cup \{(x, sin(1/x)) | x \in (0, 1]\}$ in $\mathbb{R}^2$ then to me, theorem 7.8 seems to contradict this doesn't it? How come it says that a connected open subset of $\mathbb{R}^n$ is path connected – are we saying that V is not open so theorem 7.8 isn't applicable?*
- *SELF-ASSESSMENTS*
  1. *Sheet 4 Q5ii) can we have $U = [0, 1)$ and $V = (1, 2]$ as example. If not, why not?*

– *EXAM PAPER 2007*

1. *Question 3iv): Can we have the following example to prove if X is not closed then biii) is false.*
   $S = \mathbb{R}$, $X = \mathbb{Q}$. Then X is complete since $\mathbb{R}$ is complete and $\mathbb{Q} \subset \mathbb{R}$. $\mathbb{Q}$ is not closed since Cauchy sequence $(1, 1.4, 1.14, ...)$ in $\mathbb{Q}$ tends to $\sqrt{(2)}$ not in $\mathbb{Q}$. If we have $f(x)$ is 0 if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\sqrt{2}/2$ if $x \in \mathbb{Q}$. Then f is a contraction on $(\mathbb{R}, \rho)$ and if we have $f(x) = 0$ then $x = \sqrt{2}$ holds, but $\sqrt{2}$ is not in $\mathbb{Q}$.
   Is this ok?

– CHAPTER 4

1. Yes, A is a subset of X. Almost, it even implies that $B_\delta(x)$ is contained in A^c and thus that $f(B_\delta(x))$ is contained in $f(A^c)$ and thus in $B_\varepsilon(f(x))$.
2. Almost, if $f^{-1}(F^c)$ is open, then $X \setminus f^{-1}(F^c)$ is closed (which is what we wanted to show).
   Ahh, I see that is a formulation problem: G (as subset of Y) open means that G is a neighbourhood (in Y) of any point of G. So, sorry if I wrote "neighbourhood of any point in G" instead of "neighbourhood of any point of G"!
3. Is this Theorem IV.4? Anyway, U is only a closed subset of Y if $U = \operatorname{cl} U$, the closure of any set is always closed (see Proposition III.6).
4. "For all subsets A, U", i.e., as we want (even neither open nor closed).
5. We want to show: Given f uniformly continuous, then $\operatorname{dist}(f(U), f(V)) = 0 \Rightarrow \operatorname{dist}(U, V) = 0$. By contraposition, this is proved if (given f uniformly continuous) $\operatorname{dist}(f(U), f(V)) > 0 \Rightarrow \operatorname{dist}(U, V) > 0$.
   If we have $d(x, y) < \delta$ (for some x, y), then we also have $\tilde{d}(f(x), f(y)) < \varepsilon$ (by uniform continuity) – but by the choice of x and y here we have $\tilde{d}(f(x), f(y)) = \varepsilon$, thus by contraposition again, we have $d(x, y) \geq \delta$.
   (ii)$\Rightarrow$(iii) is a similar game with contrapositive statements!
6. Because $[0, 1]$ is contained in $[0, 2]$ (or are we claiming that f is surjective?).
7. Complicated?? You surely know how to calculate maxima and minima of real (one-dimensional) function and how to calculate the derivative of $f(x)$ here, don't you?

– CHAPTER 6

1. A is an r-net for X if $\operatorname{dist}(x, A) < r$ for *all* $x \in X$!
2. Well, if there exists at least one r-net for X for every $r > 0$, then X is totally bounded. An Example where you can find exactly one finite r-net for any $0 < r < 1$, is a finite set equipped with the discrete metric!
3. Well, it says that given an(y) open covering $[0, 1]$ you can find a finite subcovering. So what is your open covering of $[0, 1]$ here? If you have such an open covering, then you should find that actually finitely many sets of this covering suffice!

– CHAPTER 7

1. Arrggh, no, I made a mistake there: the condition $G_1 \cap G_2 = \varnothing$ is only sufficient for $X \cap G_1 \cap G_2 = \varnothing$. Sorry!! If you argue with that in the exam, though, I will not deduce any points.
2. X is connected if it is not disconnected, and we have a characterisation of disconnectedness using these intersections and unions and contained/equal-things. X is thus connected if you cannot find open sets G_1, G_2 such that these properties hold, e.g., X is connected iff whenever there are open sets G_1, G_2 s.t. $G_1 \cup G_2 = X$, $X \cap G_1 \neq \varnothing$ and $X \cap G_2 = \varnothing$, then $G_1 \varnothing G_2$ is nonempty.
3. Yes, we actually show that the existence of such a continuous function onto the two-point space is equivalent to X being disconnected.
4. Why "not at all"? Okay, so it is a proof by contradiction, i.e., we assume that U is an open connected set that is not path connected. Now, the important step is to show that then every connected component U_x is open (where U_x is the set of all points of U "that can be reached by a path from x") – compare that part of the proof to the proof that any open ball is open. Now recall that the union of open sets is open. Now think about the following: What happens if you take the union of all U_s with $s \in U_x$? Well, any such s can be reached by a path from x, and any point in U_s can be reached by a path from s, but the "composition" of two paths this is again a path (by composition we mean: first "travel" through the first path then through the second, that is this $\gamma(2t)$ and $ga\tilde{m}ma(2t-1)$-thing). Hopefully, this "handwaving" argument convices you that $\bigcup_{s \in U_x} U_s$ is actually U_x itself (that is our A_1 – of course this step with the union was unnecessary, we can just set $A_1 = U_x$, but maybe/hopefully it helps to understand what is going on here). Similarly, take the union over all z not in U_x, which yields A_2 and is again an open set. Now, the catch is that these open sets are nonempty (clearly, $x \in A_1$ and by "textitnot path connected" A_2 is also nonempty). The contradiction now arises since one can show that $A_1 \cap A_2$ is neither empty nor nonempty.
   Does this make it a bit more clear?
5. Yes, V is not open in $\mathbb{R}^2$ (clearly, you cannot find a ball of any radius around, actually, any point of V contained entirely in V). A similar example was actually in the *revision voting lecture thingy*, see the Powerpoint-file on page 15 (the question starts with "The open subsets of the subspace $\mathbb{R} \times \{0\}$ of $\mathbb{R}^2 \ldots$").

– SELF-ASSESSMENTS

1. Yes, you can also take that (U', V' are the same as in the solution there then).

– EXAM PAPER 2007

1. (Of course, it should read "Then S is complete since $\mathbb{R}$ is complete and …")
   I don't see why your f should be a contraction, in fact it isn't (a very sketchy argument: any open interval in $\mathbb{R}$, say of length ε, contains both rational and irrational numbers, let x be a rational number and y an irrational number in this interval. Then, $\rho(x, y) < \varepsilon$ but $\rho(f(x), f(y)) =$

$\sqrt(2)/2$. Noting that as $\epsilon \to 0$ we still can find such x and y, f cannot be a contraction). I also do not see how from $f(x) = 0$ you get $x = \sqrt{2}$. A modification of your example $S = \mathbb{R}$, $X = \mathbb{Q}$, namely $S = [\sqrt{2}, \infty)$ and $X = \mathbb{Q} \cap [\sqrt{2}, \infty)$, works if you take the Heron-map (for finding $\sqrt{2}$), see Question 2 on Exercise sheet 8. There, however, it is also enough to take $X = (\sqrt{2}, \infty)$ (instead of additionally intersect it with $\mathbb{Q}$ also).

- *(Continuation of the previous question)* **10.1.**

  - *CHAPTER 4*

    2. *Oh ok, so G is a neighbourhood (of some points in Y) and so is a neighbourhood of any point in G itself since G is contained in Y?*
    6. *No we weren't claiming f was surjective. So it is ok, but if f was supposed to be surjective, then the mapping has to be with Y=[0,1]?*
    7. *Well I found it was complicated, but I now know how to get the* 1 *but for the part where we tend x to infinity, do we have* $f'(x) = \exp(x)(\cos(x) - \sin(x))$. *If we solve this equal to* 0 *then all I get is* $\exp(x) = 0$ *or* $\cos(x) = \sin(x)$ *and we don't (well I think we don't) want* $\exp(x) = 0$ *so* $\cos(x) = \sin(x)$ *and this happens at* $-\pi/4$ *and* $\pi/4$ *so then because we are searching for a lower bound on the function then we take* $-\pi/4$ *(not* $3\pi/4$*)? And then as far as I can tell, this is* $f(-\pi/4) = \exp(-\pi/4)\cos(-\pi/4)$ *and* $\cos(-\pi/4) = -\sqrt{2}/2$ *then the lower bound for* $f(x)$ *where* $x \in (-\infty, 0)$ *is* $-(\sqrt{2}/2)\exp(-\pi/4)$ *not* $-(1/\sqrt{2})\exp(3\pi/4)$*???*

  - *CHAPTER 6*

    2. *Ok, so we can take any finite set with the discrete metric and it only has* 1 *finite r-net as long as* $0 < r < 1$*?*
    3. *Is the open covering the union of all* $n \in \mathbb{N}$ *of* $(0, 1 - 1/n)$*???*

  - CHAPTER 4

    2. Yes, an open set is always a neighbourhood of any of its point.
    6. Yeap.
    7. Ahh, a minus sign is missing. Let me explain: You are right, we have to solve $f'(x) = 0$ and that happens exactly if $cos(x) = sin(x)$ and that happens for $x = \pi/4 + 2k\pi$ and $x = -3\pi/4 + 2k\pi$ (by the 2π-periodicity of sine and cosine), where k is an integer. So, the maxima/minima of f occur for these x. For $x < 0$ (noting that $f(0) = 1$ and $|f(x)| < 1$ for $x < 0$) we see that we are looking for the minima, which occur for the values $x = -3\pi/4 + 2k\pi$ and the "global minimum" occurs for $x = -3\pi/4$ (the exponenential "damps" the remaining minima) and we get the lower bound $-(1/\sqrt{2})\exp(-3\pi/4)$.

  - CHAPTER 6

    2. Exactly in that case!
    3. No, because you can neither "cover" 0 nor 1 with these sets! And, secondly, of course, there are many open coverings of $[0, 1]$, e.g., the union of the balls $B_{1/997}(y)$ with $y \in \mathbb{Q} \cap [0, 1]$ or $B_{1/(y^2+543)}(y)$ with $y \in \mathbb{R} \setminus \mathbb{Q} \cap [0, 1]$ or

.... Now find a finite subcoverings (hint: in the first case it is "basically the same" as finding a finite 1/997-net (with points restricted to $\mathbb{Q} \cap [0,1]$, e.g., the sets $\{B_{1/997}(k/998) \mid k \in \mathbb{Z}, 0 \leq k \leq 998\}$ will certainly be such a finite subcovering), in the second case, for example, a finite 1/543-net "located" on $\mathbb{R} \setminus \mathbb{Q} \cap [0,1]$ will do).

- *I am (somewhat worryingly), having trouble seeing why $B_{1/(x+1)^2}(x) = \{x\}$ under $(\mathbb{N}, d_{inv})$ in your model solutions. Could you please explain?* 6.1.

  Yes, I can: What (are) the nearest neighbours of x (and by "nearest neighbour" I mean the point (not equal to x itself, of course) closest to x in this metric)? I hope you see that this must either be $x+1$ or $x-1$. Now calculate the distance: We have $d_{\text{inv}}(x, x+1) = \frac{1}{x} - \frac{1}{x+1} = \frac{1}{x\cdot(x+1)}$ and, similarly, $d_{\text{inv}}(x, x+1) = \frac{1}{x-1} - \frac{1}{x} = \frac{1}{x\cdot(x-1)}$. Both numbers are greater than $\frac{1}{(x+1)^2}$, thus $B_{1/(x+1)^2}(x) = \{x\}$.

- *I have a few questions about the lecture material.* **9.1.**

  1. *In the proof that the second derived set is contained in the first derived set are we showing that $(A')'$ has the property that every neighbourhood of each point contains infinitely many points of A and so must be a limit point? Because this would imply then that in theorem 3.3 x is a limit point if and only if (not just necessity as it was stated) every neighbourhood of x contains ∞-many points of A, is this correct? It seems to be the case but I might be missing a counterexample. Would it be acceptable to use a different proof, such as showing $X \setminus A'$ is open?*
  2. *In the proof of theorem 7.5 do we need to consider the emtpy set and point sets since by the definition they are not in fact intervals?*
  3. *Will we have calculators in the exam? I'm a little unnerved by the possibility of having to calculate huge numbers when solving ODE's using Banachs FPT. Will we be expected to be able to spot expansions of functions that aren't* cos, sin *and* exp*?*
  4. *To prove the triangle inequality for the Euclidean metric in $\mathbb{R}^N$ can one reason as follows: The dot product on $\mathbb{R}^N$ is an inner product, every inner product induces a norm, a norm by definition shows that the Euclidean metric satisfies* (MS4)*?*

  1. Yes, you have actually rediscovered the statements of Corollary III.4 here! And yes, it is acceptable to use different proofs (as long as they are proofs!).
  2. Depends what your definition of interval is! How we interpret intervals, they are intervals (in these two cases, of course, there is no subset (x, y) with $x < y$ and $x, y \in I$ – so, trivially, all such subsets are contained in I).
  3. No calculators, there will be no huge numbers and no expansions beyond cos, sin and exp.
  4. Yes, that's possible (of course, you then only "transport" the proof of (MS4) simply to a similar statement for the norm).

- *Please can you advise if you have the solutions to the 2002/2003 exam question part (c) that you included in the Drop-in 6 Homeomorphisms session material. I can not find them on the library website.* 7.1.

  Ha, earlier this semester the 2002/03-exam was available on the library website, but you are right, now it isn't anymore.
  Okay, so Question 1(c) on Drop-in Sheet 6: For the homeomorphism part see Exercise sheet 3 Question 4 (together with what we said about homeomorphisms and equivalent metrics at the end of Chapter IV). For the second part, note that being isometric metrics is a special case of being uniformly equivalent metrics (and also being equivalent metrics) and use an argument as in Exercise sheet 3 Question 2 to show that they have the "same" (convergent(!) in the case of completeness) Cauchy sequences.

- *After going over my notes for last week (lectures from monday 17th) I am still left confused as to the proof of Picard's Theorem. Would it be possible to outline to me here?*

  Okay, so Picard's Theorem is an application of Banach's Contraction Mapping Principle to differential equations. Thus, we need a complete metric space and a contraction acting on this space. If we have this then the existence and uniqueness of a fixed point is assured. That's the goal!

  Now, for a differential equation, this fixed point should (of course) be the solution to the differential equation. Now instead of a differential equation, we can also look at an integral equation. The nice thing about rewritting it in this integral form is that it already is a fixed point equation: the fixed point/the solution equals something, namely, a certain integral which actually involves the solution (i.e., $\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))\,\mathrm{d}s$). But we can now interpret this right-hand side as a map on functions: we input a function to the integral and get a function back. So, we now want to show that this function T (on the space of functions) is actually a contraction (i.e., we are now looking at $T(\psi)(t) = x_0 + \int_{t_0}^{t} f(s, \psi(s))\,\mathrm{d}s$).

  So, setting the scene is done; the questions we have to talk about are:

  - Contraction on what space?
  - Contraction w.r.t. which metric?

  The answer to the first is straightforward: The solution should be continuous on some interval and that interval should line up with the initial condition. In fact, the subset of continuous functions we are looking at here, is a closed subset and therefore the corresponding subspace is complete (we need completeness for Banach to work).

  Now, that (i.e., completeness, closedness of this subspace) is w.r.t. the uniform metric $d_{\max}$. However, we are not restricted to this metric: If we choose a uniformly equivalent metric, then still this subspace will be closed and complete (we have the same Cauchy sequences and convergent sequences for uniformly equivalent metrics). So, we change to such a uniformly equivalent metric (in fact, we are looking at a whole parametrized family, so we have a parameter β available we can fix later on). That has the following advantage: While our map T might not be a contraction w.r.t.

$d_{\max}$, it might be possible to choose a different (albeit still uniformly equivalent) metric such that T is a contraction (in plain words the following might happen: it might only be possible to show that $d_{\max}(T(f), T(g)) < 1.5\ d_{\max}(f, g)$ – so T is not a contraction, well w.r.t. $d_{\max}$ –, but using a uniformly equivalent metric d we actually can show that $d(T(f), T(g)) < 0.8\ d(f, g)$). That's where we use this parameter β and we then easily establish that T is a contraction!

With all this established, we let Banach do the rest of the work and Picard's Theorem is proved.

- *In Picard's Theorem, how does this* $\log(2)$*-thing arise?*

  We want to establish the following estimate $\left|e^{\beta t} - e^{\beta t_0}\right| \leq e^{\beta t}$. So, calculate:

  $$\left|e^{\beta t} - e^{\beta t_0}\right| \;=\; e^{\beta t} \cdot \left|1 - e^{\beta (t_0 - t)}\right|$$

  and the term in $|\dots|$ is less than or equal to 1 whenever $t \geq t_0 - \frac{\log(2)}{\beta}$: note that $0 < e^{\beta (t_0 - t)} \leq 1$ if $t \geq t_0$ and $e^{\beta (t_0 - t)} > 1$ if $t < t_0$ to get rid of the absolute value here. This then yields the condition $e^{\beta (t_0 - t)} \leq 2$ and thus $t \geq t_0 - \frac{\log(2)}{\beta}$ for $|\dots|$ to be less than or equal to 1.

- *A quick question on exercise sheet 9 Q4(i): do we not need to show this x_0 result to be unique – by assuming there exists an x_1 s.t. $f(x_1) = x_1$, and then show that $d(x_1, x_0) = 0$ (showing $x_0 = x_1$)? If not, why?*

  You are absolutely right, I missed that one in the model solution and your argument is correct. Thanks for pointing this out!!

- *I was just working through Exercise sheet 9 with the answers and there were a few things I wasn't sure of and thought I'd email them to you while they were fresh in my head:*

  – *Question 1(iv): This one I just don't follow the solution online, mainly the lines leading up to intersection over natural numbers of A_n not empty. Mainly confusion with the indices, sorry that is vague.*

  – *Question 3(ii): The solutions say that A_1 not closed and thus also not bounded? I don't understand this, it seems to me that A_1 is bounded?*

  – *Question 4(i): I don't understand how the argument implies g is continuous as you already know $f(x_k) - f(x)$ will tend to zero so how does $2\,d(x_k, x)$ tends to 0 help?*
    Also on this question I am not sure about the uniqueness part, why is it necessary for $d(x_0, y_0) = 0$?

  – *Question 4(ii): I don't understand how you obtain $d(f^n(\tilde{x}), f^m(\tilde{x})) < \varepsilon$ from the line above as n, m need to be bigger than n_k and N, surely bigger than N doesn't always work necessarily?*

  – *Question 6(ii): Why are the sets given as answers not compact w.r.t. jungle river metric?*

  Sorry, you hit a streak of typos here!!

- Question 1(iv): It should read: "Since $x_n \in A_N$ $\forall n \geq N$ and A_N is closed..." (instead of "Since $x_k \in A_N$ $\forall n \geq N$..."). The point here is that if you choose an element x_n from each A_n then you automatically get a Cauchy sequence (x_n); by the closedness of the A_n, the limit belongs to all A_n and thus also to the intersection (and this intersection actually contains only this limit, i.e., whatever x_ns you choose, the limit is always the same).
- Question 3(ii): It should read compact instead of bounded. Sorry!
- Question 4(i): The first line in the chain of inequalities should read: "$g(x_k) - g(x) = d(x_k, f(x_k)) - d(x, f(x)) \leq \ldots$" (and not "$f(x_k) - f(x) = \ldots$"). Then, following the same steps for $g(x) - g(x_k)$ one establishes that $|g(x_k) - g(x)| \leq 2\,d(x_k, x) \to 0$ which establishes the continuity of g.
  On the uniqueness, note that $d(x_0, y_0) = d(f(x_0), f(y_0))$ since x_0, y_0 are fixed points, however, $d(x_0, y_0) > d(f(x_0), f(y_0))$ if x_0 not equal to y_0 since f is contractive. Thus $x_0 = y_0$.
- Question 4(ii): Okay, let us restart with at the sentence starting with "But...": "But $d(f^n(\tilde{x}), y) \leq d(f^{n_1}(\tilde{x}), y)$ for all $n \geq n_1$ (by the definition of "contractive"), thus $d(f^n(\tilde{x}), f^m(\tilde{x})) \leq \ldots < \varepsilon$ for all $n, m \geq n_1, \ldots$"
  So, I simply choose the number n_1 to establish that $(f^n(\tilde{x}))$ itself is Cauchy.
- Question 6(ii): Use the first criterion in Question 6(i) ("$\forall \varepsilon > 0$ the sets ... are finite") and observe that $[0, 1]$ (or any proper interval in $\mathbb{R}$) is uncountable (and in particular infinite).

- *In the lecture you have shown that $\mathbb{Q}$ as subspace(!) of $\mathbb{R}$ is disconnected. What if you do not treat $\mathbb{Q}$ as subspace but as metric space of its own right?*

  So, to check if $\mathbb{Q}$ (as metric space itself) is connected or disconnected, let us check if we can find nontrivial clopen subset of $\mathbb{Q}$. Using that open balls are open and closed balls are closed, you can easily check that $B_{\sqrt{2}}(0) = \{x \in \mathbb{Q} \mid |x| < \sqrt{2}\}$ (the open ball of radius $\sqrt{2}$ around 0 in $\mathbb{Q}$) and $\overline{B}_{\sqrt{2}}(0) = \{x \in \mathbb{Q} \mid |x| \leq \sqrt{2}\}$ (the closed ball of radius $\sqrt{2}$ around 0 in $\mathbb{Q}$) are equal as sets and therefore this set is clopen. Consequently, there is a nontrivial clopen set, thus $\mathbb{Q}$ is disconnected.

  So, this proof does not use that $\mathbb{Q}$ is a subset (and thus a subspace) of $\mathbb{R}$. However, the reason why here it does not matter wether we regard $\mathbb{Q}$ as metric space of its own right or "only" as a subspace of $\mathbb{R}$, is actually the completion theorem: $\mathbb{R}$ is the (essentially) unique completion of $\mathbb{Q}$ (now check how/that (dis)connectedness in a space implies (dis)connectedness in its completion).

  You might wonder, though, why $\mathbb{Q}$ is not path connected. However, check that there is no non-constant continuous function from $[0, 1]$ to $\mathbb{Q}$ (basically, observe that $\mathbb{Q}$ is a discrete space, i.e., all subsets are clopen, then use the characterisation of continuity and that $[0, 1]$ (an interval!) is connected; also compare to Proposition VII.3 and the proof of Theorem VII.7).

- *Please can you advise on how to complete the following to show that "this is immediately established using the definition of continuity and the product metric in" in Problem Sheet 10 Q1(iv).*

With ε-δ-criterion: We know that $\gamma : X \to \mathbb{R}$ and $\tilde{\gamma} : Y \to \mathbb{R}$ are continuous, i.e., for every $\varepsilon > 0$ there exist $\delta_1 > 0$ s.t. $|r - s| < \delta_1$ implies $d(\gamma(r), \gamma(s)) < \varepsilon$ and $\delta_2 > 0$ s.t. $|r - s| < \delta_2$ implies $\tilde{d}(\tilde{\gamma}(r), \tilde{\gamma}(s)) < \varepsilon$. Set $\delta = \min\{\delta_1, \delta_2\}$. Then, $|r - s| < \delta$ implies – using the max-metric – that $d_{\max}((\gamma, \tilde{\gamma})(s), (\gamma, \tilde{\gamma})(r)) = \max\{d(\gamma(r), \gamma(s)), \tilde{d}(\tilde{\gamma}(r), \tilde{\gamma}(s))\} < \varepsilon$. Thus $(\gamma, \tilde{\gamma})$ is continuous.

To summarise: for every $\varepsilon > 0$ we find, by the continuity of γ and $\tilde{\gamma}$, numbers $\delta_1, \delta_2 > 0$. But by using the minimum of these two numbers (so, there exists a δ, namely the minimum of δ_1 and δ_2, s.t. ...), we establish the continuity in $X \times Y$.

Questions related to and/or on the upcoming final exam:

- *Do we need to be able to recall specific metrics for the examination? The more obscure ones like the Jungle River Metric etc.*

  The “more obscure” ones will be given in the question text (e.g., as in Question 2(ii) on Exercise sheet 7). You should be familiar, though, with the discrete metric on any space, the Euclidean metric and the product metrics d_1 and d_∞ on $\mathbb{R}^n$ and the uniform metric on $C[a, b]$.

- *I was wondering whether the proof of Theorem I.3 and Proposition I.4 from chapter one from the lecture notes for MA30041 are examinable? They have been given as exercises, but should we consider them for the exam in January?*

  Sure, things done in the exercise sheets are examinable.

- *On Theorem II.10: just checking that since we never did an exact proof for this theorem and we only outlined the process. Does this mean that this proof is too hard for us to be examined on?*

  Simply: yes! (Or, well, I don’t want to imply that it is too hard for you, but you will not be examined on it.)

- *I can’t remember what you said in the lecture, is the proof (which was given as a handout) of Theorem VII.5 examinable?*

  Yes, it is the one exception to all handouts, it is actually examinable!

- *I’ve been doing some past exam papers and am having trouble when trying to prove* **9.1.** *theorems and lemmas. Is there a way to start these questions or a structure to follow for questions like these?*

  If I would know an answer to that question I would have proved the Riemann hypothesis;-)
  Okay, without joking, there is of course no general answer to “how do I prove something”. However, in an exam situation, I guess, a good strategy is to (after a deep breath) start with the definitions of the things stated in the lemma/theorem to be proved (e.g., if the assumption is that “X has property A” and the claim is that “Y has property B”, write down the definition of A and B – often they are asked

in earlier parts of the same exam question). Then, recall alternative statements (e.g., what is an equivalent formulation of ”property A/B”), or statements that are implied (e.g., “A implies C”, and maybe you discover that “C implies ‘Y has B’ ” and you are done) or statements that the contrapositive implies (e.g., “if Y does not have property B, then it has property C” and you might discover that this implies that “X has not property A” and you are again done – this time by a proof by contraposition). And so on.... Of course, finding these connections “efficiently” is down to experience (lots of practice) and mathematical intuition.

Best, and good intuition on Monday!

- *I am a student of yours, at metric spaces (MA30041), and I am missing the lecture notes from week 3 & 4, and I don't know If I could get them tomorrow from any of my fellow students, is there any chance that you have copies of these two lectures,which I can take sometime tomorrow?*

  You can find a detailed list of things we have done in the lectures as “overview of topics we have covered so far” (with the latest and last version of 1 Dec) on the course website and/or the Moodle page. It contains appropriate (exact) references to the (e)textbook we are using in this course. Other than that, I am afraid, I do not have copies of these two lectures for you.

- *Are you going to upload the complete lecture notes onto moodle at the end of the course? I know it is early for such preperation work on my part, but I am finding the module particularly difficult, and have always found electronic notes help my revision hugely.*

  No, I am not planing to upload lecture notes onto Moodle/internet. That is why we have this e-textbook and the regularly updated overview what we have done in the lectures (referencing the textbook).

  I do not really understand what you mean by “I know it is early for such preperation work on my part”? That's why I am putting previous exam questions and self-assessment sheets already now on the Moodle/internet-page of this unit.

- *Is there material covered in lectures which is not covered in the e-book?*

  Yes, but then it is covered by a question on an exercise sheet.

- *I was wondering if it possible that you may have the complete set of lectures notes available online. Thanks.* 31.12.

  I can only repeat what I said before (you can find these answers in the FAQ-file `faq.pdf` on the course website/Moodle): I am not uploading lecture notes onto Moodle/internet, in fact, I even do not have a complete set of lecture notes in any electronic form. That is why we have this e-textbook and the regularly updated overview what we have done in the lectures (referencing the textbook).

- *To what extent will you be following the format/structure of previous exam papers this year? (I noted the problem sheets are different from last year)*

  Uhm, what do you expect me to say here. Of course, the format/structure is “the same”: you have 2 hours to answer 3 out of 4 questions. I will check if you know

the definitions, some results and how to prove them, and there will be some unseen bits.
But maybe (only maybe!) you are actually asking a quite different question, namely: is it enough to memorize all the stuff in the previous exam papers and not bother with all these exercises to get $xy\%$ (with x greater than 4 or 5 or 6)? Then, of course, the answer has to be NO whatever number you pick (if I have any self-respect for my teaching!).

- *Since you are a new lecturer to this module (at Bath!) many of us are unsure as to what to expect in the exam papers – since your course notes are mostly quite different from Toland's last year also.*

  I am also excited about the exam (well, not about the actual marking part, but that is a different story...)! I disagree with the statement "mostly quite different", though.

- *I've been looking through a lot of the past exams and it looks like Prof Toland has made his course very definition and example heavy, and from the looks of your tutorial sheets there seems to be a fair bit more proof on your side, would the exam be a reflection of this? If you think the answer to this is a bit to close to asking exactly what will be on the exam then please feel free not to answer it!*

  Well, exercise sheets, of course, reflect the personal "taste" of the lecturer (so here, what do I think is "important to understand"). Will I test understanding in the exam? At least, I like to think so. Does this now mean "more proofs" in the exam? Hmm, maybe, maybe not – and maybe I am just the wrong (especially, not very objective) person to answer that question (one issue: how do you discriminate an example from a proof?). Sorry.

- *Since a few of us have a clash with the drop-in session and another module's lectures, if we have any problems now we are coming to properly revise the material will we be able to organise a time to come and see you?*
  It is getting to the point in the year where we start panicking about revision and exam, and as I am sure you are aware Metric Spaces has alot of material in comparison to other modules!

  No problem, organise a time! As I said right at the beginning of this course (and as it is stated on the course website/Moodle page), if you have questions speak to me after the lecture/exercise session or make an appointment. And if there are a few of you, we will certainly find a time (a maybe even a room) to discus your questions.

  I am trying to give you as much support as possible; that is why there are all these additional materials available on the course-web/Moodle-site. However, it is your responsibility to engage with the material (and maybe try to hand in solutions to some of the questions on the exercise sheet).

- *I'm just wondering if you are going to be holding a revision session for this module in January?*

  Right now I have no plans to hold another revision session in January.