MA30041: Metric Spaces

Old Exams 7: Contractions

1.) *From the 2006/07-exam:*

(a) What is a *contraction mapping* on a metric space (X, d)?

(b) State, without proof, *Banach's contraction mapping principle.*

(c) Suppose that (X, d) is complete, $f : X \to X$ is a contraction mapping and that $A \subset X$ is nonempty and closed in (X, d) with $f(A) \subset A$. Show that A contains the fixed point of f.

(d) Show, by example, that the conclusion of (c) may be false if A is not closed.

2.) *From the 2003/04-exam:*

Let $X = C[0, 1]$ equipped with the uniform metric $d_{\max}$. Show that $T : X \to X$ defined by

$$T(f)(x) = \frac{1}{2}\left(\int_0^x f(t)\,\mathrm{d}t\right) + 1$$

for $x \in [0, 1]$ and all $f \in X$, defines a contraction on $(X, d_{\max})$. Determine the limit of the sequence $(f_n) \subset X$ defined by

$$f_1 \equiv 0, \quad f_{n+1} = T(f_n) \forall n \in \mathbb{N}.$$