MA30041: Metric Spaces

OLD EXAMS 5: CONTINUOUS FUNCTIONS

1.) *From the 2003/04-exam:*

Let (X, d), $(\tilde{X}, \tilde{d})$ be metric spaces.

(a) What is meant by saying that

(i) $U \subset X$ is *open* in (X, d),

(ii) $f : X \to \tilde{X}$ is *continuous* at $x_0 \in X$,

(iii) $f : X \to \tilde{X}$ is a *homeomorphism*?

(b) Prove that the open ball $B_\varepsilon(x_0)$ of radius $\varepsilon > 0$ and centre $x_0 \in X$ is an open set.

(c) Prove that $f : X \to \tilde{X}$ is continuous at $x_0 \in X$ iff $f(x_n) \to f(x_0)$ as $n \to \infty$ in $(\tilde{X}, \tilde{d})$, for any sequences $x_n \to x_0$ in (X, d).

(d) Prove that f is continuous on (X, d) iff $f^{-1}(U)$ is open in (X, d) for any open set U in $(\tilde{X}, \tilde{d})$.

(e) Let $X, \tilde{X} \subset \mathbb{R}$, $X = [0, 1] \cup (2, \infty)$, $\tilde{X} = [0, \infty)$ be equipped with the subspace metrics induced by the usual metric on $\mathbb{R}$. Define $f : X \to \tilde{X}$ by

$$f(x) = \begin{cases} 4x^2 & \text{if } x \in [0, 1], \\ x^2 & \text{if } x \in (2, \infty). \end{cases}$$

Then f is a bijective continuous map from X to $\tilde{X}$. Is f a homeomorphism? (Justify your answer.)