MA30041: Metric Spaces

On the "Completion-Theorem" (Not Examinable!)

Theorem II.10 ("Up to isometry, every metric space has a unique completion")
Every metric space (X, d) *has a* completion, *i.e.,* (X, d) *is a metric subspace of a complete metric space* (X^*, d^*) *and for every* $x \in X^*$ *there exists a sequence* $(x_n) \subset X$ *s.t.* $\lim_{n\to\infty} x_n = x$. *Moreover, any two completions are isometric to each other.*

Remarks on the Proof:

- Set $\hat{X} = \{(x_n) \subset X \mid (x_n) \text{ is a Cauchy sequence } \}$.
- Define a pseudometric ρ on $\hat{X}$ by

  $$\rho\left((x_n), (y_n)\right) = \lim_{n\to\infty} d(x_n, y_n)$$

  Note: Since $(x_n), (y_n) \subset X$ are Cauchy, so is $(d(x_n, y_n)) \subset \mathbb{R}$. But every Cauchy sequence in $\mathbb{R}$ is convergent, thus this limit exists.
- As in Exercise sheet 2 Question 2, define an equivalence relation "$\sim$": $(x_n) \sim (y_n)$ iff $\rho\left((x_n), (y_n)\right) = 0$.
  $\rightsquigarrow$ a metric $\tilde{d}$ on $\tilde{X} = \hat{X}/\sim$
  Note: Equivalence class are all Cauchy sequences that converge to the same "limit".
- Note: $\phi : X \to \tilde{X}$, $x \mapsto [(x, x, x, x, \ldots)]$ is an isometry from (X, d) to $(\tilde{X}, \tilde{d})$, thus (X, d) is a metric subspace of $(\tilde{X}, \tilde{d})$.
- We have (by construction): Any Cauchy seuqence $(x_n) \subset X$ has a limit in $\tilde{X}$.

  Problem: Is $(\tilde{X}, \tilde{d})$ complete?

  So, we have to consider Cauchy sequences in $\tilde{X}$ (which is the space of equivalence classes of Cauchy seuqences of X!).
  This part of the proof is very technical and does not become clearer if someone explains it (one has to carefully work through it by oneselfs). The catch is that one does not get "anything new" by considering Cauchy sequences in $\tilde{X}$!
- The proof of the uniqueness makes use of the definition of "completion": The completion of a metric space (X, d) is the "minimal possible" complete superspace.
- For details see the textbook (*Shirali et al.*): Proof of Theorem 1.5.3 (pp. 55ff).