MA30041: Metric Spaces

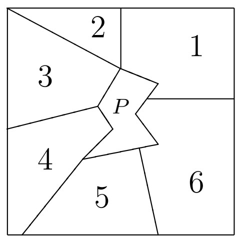
Five Challenging Problems

For each of the following problems, a prize of GBP 20.– (in vouchers for the Parade bar or the book shop) will be awarded for the first correct (and sufficiently justified) solution handed in by a student registered for MA30041/50182 “Metric Spaces” in the academic year 2008-09 on or before 9 January 2009. Please note that Bernd Sing did not invent these problems (they are “well-known”) and the solutions might or might not have anything to do with MA30041/50182 “Metric Spaces”.

1.) (a) Show that for any integer $N > 2$ it is possible to find N points in the plane, not all on one line, such that the (Euclidean) distance between any two of them is an integer.

(b) Show that it is impossible to find infinitely many points in the plane satisfying the conditions of part (a).

2.) A square of sidelength 1 is divided into polygons. Suppose that each of these polygons has a diameter (w.r.t. the Euclidean metric) less than $\frac{1}{30}$. Show that there is a polygon P with at least six neighbours, i.e., polygons touching P in at least one point.



3.) Give (with proof) an explicit example of a positive real number α such that the sequence $(\cos(n!\,\alpha))$ does not converge.

4.) Let A be a subset of a metric space (X, d). Consider the collection of all subsets of X which can be obtained from A by taking successively either the complement in X or the closure (e.g., A, $X \setminus A$, $\operatorname{cl} A$,$\operatorname{cl}(X \setminus A)$, etc.). Show that no more than 14 of these sets may be different from each other. Show also that it is possible to obtain 14 sets when $X = \mathbb{R}$ (with the usual metric) if A is a suitable subset of $\mathbb{R}$.

5.) Is there a continuous function $f : [0, 1] \to \mathbb{R}$ with the following property: for all $x \in [0, 1]$, $f(x)$ is rational iff x is irrational?

Reminder: Deadline in the essay-writing competion is Tuesday November 11!