MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 9

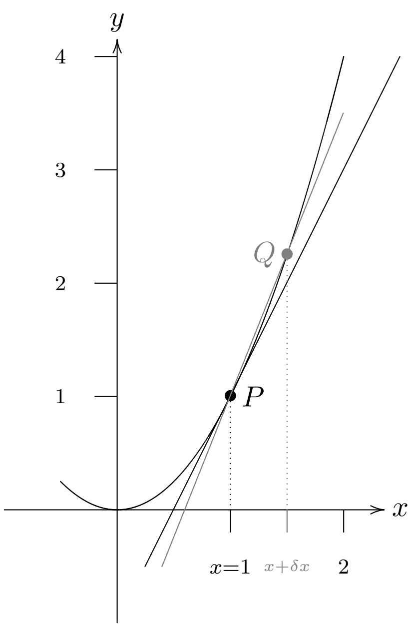
§9 Differentiation

We have previously defined the gradient of a straight line. What about the gradient of a more general curve? In general the gradient will change as you move along the curve.

Consider a point $P(x, y)$ on the curve and let Q be a nearby point on the curve with x coordinate $x + \delta x$, where δx is "small" (so δx is one symbol, "δ" is the lower case delta and here stands for "small difference in the x-coordinate"). To find the gradient at P we find the gradient of PQ and then let δx get arbitrarily small. More precisely we take the limit of the gradient of PQ as $\delta x \to 0$.

The result of this procedure is the the gradient of the tangent line at P, i.e., the gradient of a curve in a point P is the gradient of the tangent to that curve at the point P.

EXAMPLE: If $y = x^2$, find the gradient of the tangent line at $P(1, 1)$.



The problem is how one can find the gradient of the tangent at $P(1, 1)$. So, for the nearby points $Q(x_0, x_0^2)$ on the curve we compute the following table:

x_0	x_0^2	Q	gradient of PQ
2	4	$(2, 4)$	3
1.5	2.25	$(1.5, 2.25)$	2.5
1.1	1.21	$(1.1, 1.21)$	2.1
1.01	1.0201	$(1.01, 1.0201)$	2.01
0.5	0.25	$(0.5, 0.25)$	1.5
0.9	0.81	$(0.9, 0.81)$	1.9
0.99	0.9801	$(0.99, 0.9801)$	1.99

From this table, we might guess that the gradient at $x = 1$ is 2.

We now consider the general case for $y = x^2$ (i.e., not only at one specific point).

EXAMPLE: If $y = x^2$ find the gradient of the tangent at any point P on the curve.
A general point P has coordinates (x, y) with $y = x^2$.
If Q is the point with $x-$coordinate $x + \delta x$, then its $y-$coordinate is $y + \delta y = (x + \delta x)^2$.
and the gradient of the line PQ is

$$\begin{aligned}
\frac{\delta y}{\delta x} = \frac{\text{difference of } y\text{-coordinates}}{\text{difference of } x\text{-coordinates}} &= \frac{(x+\delta x)^2 - x^2}{\delta x} \\
&= \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x} = \frac{2x\delta x + (\delta x)^2}{\delta x} \qquad (1) \\
&= 2x + \delta x \qquad (2) \\
&\to 2x \quad \text{as} \quad \delta x \to 0.
\end{aligned}$$

And indeed, we see that the tangent at $x = 1$ has gradient 2 (while the tangent at $x = 2$ has gradient $2 \times 2 = 4$ and the one at $x = 0$ has gradient 0).

NOTE: We cannot put $\delta x = 0$ directly in (1), since this would give the undefined expression $0/0$. But when we arrive at (2), we have divided out the troublesome δx on the bottom, and we can calculate the limit of $2x + \delta x$ as $\delta x \to 0$ by directly putting δx equal to 0.

DEFINITION: The limit of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$ is called the *derivative* of y with respect to x, denoted $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}}{\mathrm{d}x} y$. If we write y as function of x, i.e., $y(x) = x^2$, then also the notation $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ is used.
The process of computing the derivative is called *differentiation*.

Derivative of x^3
Repeating the argument for x^2 leads to:

$$\begin{aligned}
\frac{\delta y}{\delta x} = \frac{(x+\delta x)^3 - x^3}{\delta x} &= \frac{x^3 + 3\,x^2\,\delta x + 3\,x\,(\delta x)^2 + (\delta x)^3 - x^3}{\delta x} \\
&= \frac{3\,x^2\,\delta x + 3\,x\,(\delta x)^2 + (\delta x)^3}{\delta x} \\
&= 3\,x^2 + 3\,x\,\delta x + (\delta x)^2 \\
&\to 3x^2 \quad \text{as} \quad \delta x \to 0.
\end{aligned}$$

So we have found that

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\,x \qquad \text{when} \qquad y = x^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\,x^2 \qquad \text{when} \qquad y = x^3
\end{aligned}$$

The general formula for differentiating x^a is

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\, x^a \;=\; a\, x^{a-1}}$$

This holds for any $a \neq 0$ (not just integers)

Two other ways of writing this are:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= a\, x^{a-1} \qquad \text{when} \qquad y = x^a, \qquad \text{or} \\ f'(x) &= a\, x^{a-1} \qquad \text{when} \qquad f(x) = x^a. \end{aligned}$$

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We have derived the following rule:

$$\boxed{\text{Rule 1:} \quad \frac{\mathrm{d}}{\mathrm{d}x}\, x^a \;=\; a\, x^{a-1}}$$

We now state some further rules (more will come next week) for differentiating:

$$\boxed{\text{Rule 2:} \quad \frac{\mathrm{d}}{\mathrm{d}x}\,(a\, f(x)) \;=\; a\,\frac{\mathrm{d}}{\mathrm{d}x}\, f(x) \;=\; a\, f'(x)}$$

for any constant a.

We give the reason for this rule: By definition we have $\frac{f(x+\delta x)-f(x)}{\delta x} \to \frac{\mathrm{d}}{\mathrm{d}x}\, f(x)$ as $\delta x \to 0$. But multiplying the function f by a number a, we have

$$\frac{a\, f(x+\delta x) - a\, f(x)}{\delta x} = a\,\frac{f(x+\delta x) - f(x)}{\delta x}$$

and the right-hand side goes to $a\,\frac{\mathrm{d}}{\mathrm{d}x}\, f(x)$ as $\delta x \to 0$.

Similarly, one can show the next rule.

$$\boxed{\text{Rule 3:} \quad \frac{\mathrm{d}}{\mathrm{d}x}\,(f(x)+g(x)) \;=\; \frac{\mathrm{d}}{\mathrm{d}x}\, f(x) + \frac{\mathrm{d}}{\mathrm{d}x}\, g(x) \;=\; f'(x)+g'(x)}$$

So, the derivative of a sum of functions is the sum of the derivatives.

We already know the last rule: The constant function $y = a$ is the line parallel to the x-axis and therefore has gradient 0. Using the words of differentiation, we therefore have:

$$\boxed{\text{Rule 4:} \quad \frac{\mathrm{d}}{\mathrm{d}x}\, a \;=\; 0}$$

for any constant a.

Examples

(a) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3+3x+7\right) \overset{\text{Rules 3\&2}}{=} 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\, x^3\right) + 3\left(\frac{\mathrm{d}}{\mathrm{d}x}\, x\right) + \left(\frac{\mathrm{d}}{\mathrm{d}x}\, 7\right)$$
$$\overset{\text{Rules 1\&4}}{=} 2 \times 3\, x^2 + 3 \times 1\, x^0 + 0 = 6\, x^2 + 3.$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(3\sqrt{x} + 2(\sqrt{x})^3\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(3\, x^{1/2} + 2\, x^{3/2}\right)$$
$$= 3 \times \frac{1}{2} x^{-1/2} + 2 \times \frac{3}{2}\, x^{1/2} = \frac{3}{2\sqrt{x}} + 3\sqrt{x}.$$

(c) $$\frac{\mathrm{d}}{\mathrm{d}x}\left((x+2)^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 4\, x + 4\right) = 2\, x + 4.$$

(d) $$\frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)\,(x+2)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 3\, x + 2\right) = 2\, x + 3.$$

Interpretation of the derivative

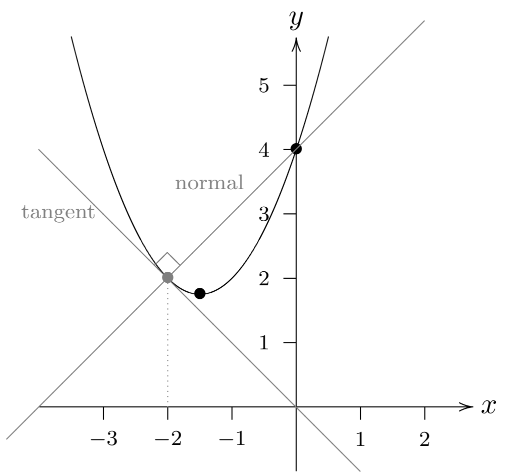
If $f'(x)$ is positive then f is increasing at x ("gradient points up").
If $f'(x)$ is negative then f is decreasing at x ("gradient points down").
If $f'(x) = 0$, then f is *stationary* at x
(may be a max or min; gradient is horizontal).

EXAMPLE: Find where $f(x) = x^2 + 3x + 4$ is increasing, stationary and decreasing. Sketch the curve $y = f(x)$. Also find the equation of the tangent and the *normal* (i.e., the line perpendicular to the tangent) at $x = -2$.

Differentiating $f(x)$ yields $f'(x) = 2\,x + 3$.
So, $f(x)$ is increasing when $2\,x + 3 > 0$, i.e., when $x > -\frac{3}{2}$.
It is decreasing when $2\,x + 3 < 0$, i.e., when $x < -\frac{3}{2}$.
And it is stationary when $x = -\frac{3}{2}$.

The value of f at the stationary point is $f(-\frac{3}{2}) = \frac{7}{4} = 1.75$. So, the point $\left(-\frac{3}{2}, \frac{7}{4}\right)$ is on the curve. Actually, it is the minimum, since f is decreasing left of this point and increasing on the right of this point. Here we see that instead of completing the square to calculate the minimum, one can also calculate and analyse the derivative.

Since the minimum is above the x-axis, there are no x-intercepts. The y-intercept is $(0, f(0))$ which is $(0, 4)$.



At $x = -2$ we have $f(-2) = 2$ and $f'(-2) = -1$. Therefore, the tangent passes through $(-2, 2)$ and has gradient -1. This yields the equation $y = -x$ for the tangent (it also passes through the origin).
The gradient of the normal (which also passes through (-2,2)) is 1 (since the product of the gradients of the tangent and the normal is -1) wherefore we obtain the equation $y = x + 4$ (the normal also passes through the y-intercept of $y = f(x)$).

A sketch of $y = f(x)$ (with tangent and normal at $x = -2$ in gray) can be found on the left.

END OF LECTURE 18