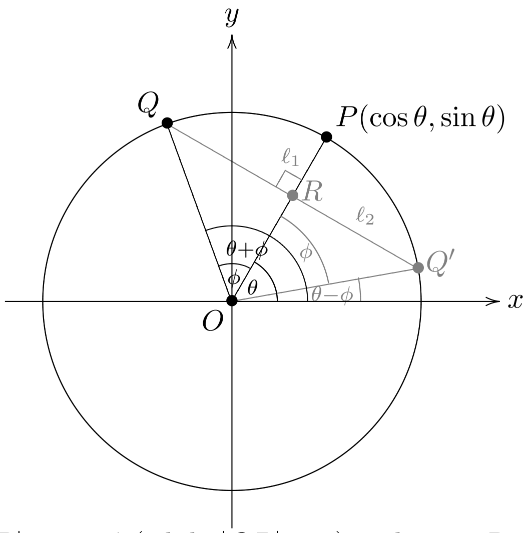
MA10103: Foundation Mathematics I

Lecture Notes – Week 8

Compound angle formulae (calculations not lectured, derivation not examinable)

We recall the definition of $\sin\theta$ and $\cos\theta$ as coordinates of the point P on the unit circle, see figure to the right.



Consequently, the point Q has coordinates $(\cos(\theta+\phi), \sin(\theta+\phi))$. However, we can also calculate the coordinates of Q as follows: Q is (one of) the intersection point(s) of the line ℓ_2 with the unit circle $x^2 + y^2 = 1$. But the line ℓ_2 is perpendicular to the line ℓ_1 where ℓ_1 is the line through O and P, and ℓ_1 and ℓ_2 intersect in the point R.

We calculate the coordinates of R: We have $|OR| = \cos\phi$ (while $|OP| = 1$) and since R is on ℓ_1 and between O and P we immediately have

$$R(\cos\theta\,\cos\phi, \sin\theta\,\cos\phi).$$

The gradient of ℓ_1 is $\tan\theta = \frac{\sin\theta}{\cos\theta}$, thus ℓ_2 has the equation

$$\begin{aligned} y &= -\frac{\cos\theta}{\sin\theta}\,(x - \cos\theta\,\cos\phi) + \sin\theta\,\cos\phi \\ &= -\frac{\cos\theta}{\sin\theta}\,x + \frac{\sin^2\theta + \cos^2\theta}{\sin\theta}\,\cos\phi \\ &= -\frac{\cos\theta}{\sin\theta}\,x + \frac{\cos\phi}{\sin\theta}. \end{aligned}$$

We can now calculate the intersection points of ℓ_2 and the unit circle: One of them is $Q(\cos(\theta+\phi), \sin(\theta+\phi))$, the other one is $Q'(\cos(\theta-\phi), \sin(\theta-\phi))$. Indeed, we get a quadratic equation for the x-coordinate, namely

$$\begin{aligned} 1 &= x^2 + y^2 \\ &= x^2 + \left(-\frac{\cos\theta}{\sin\theta}\,x + \frac{\cos\phi}{\sin\theta}\right)^2 \\ &= \frac{\sin^2\theta}{\sin^2\theta}\,x^2 + \frac{\cos^2\theta}{\sin^2\theta}\,x^2 - 2\frac{\cos\theta\,\cos\phi}{\sin^2\theta} + \frac{\cos^2\phi}{\sin^2\theta} \\ &= \frac{x^2 - 2\,\cos\theta\,\cos\phi\,x + \cos^2\phi}{\sin^2\theta}, \end{aligned}$$

which can be rewritten as

$$x^2 - 2\cos\theta\,\cos\phi\,x + (\cos^2\phi - \sin^2\theta) = 0.$$

The solution for this quadratic equation is

$$\begin{aligned}
x &= \frac{2\cos\theta\,\cos\phi \pm \sqrt{4\cos^2\theta\,\cos^2\phi - 4(\cos^2\phi - \sin^2\theta)}}{2} \\
&= \cos\theta\,\cos\phi \pm \sqrt{\cos^2\theta\,\cos^2\phi - \cos^2\phi + \sin^2\theta} \\
&= \cos\theta\,\cos\phi \pm \sqrt{\cos^2\phi\,(\cos^2\theta - 1) + \sin^2\theta} \\
&= \cos\theta\,\cos\phi \pm \sqrt{-\cos^2\phi\,\sin^2\theta + \sin^2\theta} \\
&= \cos\theta\,\cos\phi \pm \sqrt{(-\cos^2\phi + 1)\,\sin^2\theta} \\
&= \cos\theta\,\cos\phi \pm \sqrt{\sin^2\phi\,\sin^2\theta} \\
&= \cos\theta\,\cos\phi \pm \sin\phi\,\sin\theta.
\end{aligned}$$

Noting that in the case $\theta = \phi$ the x-coordinate of Q' is $\cos 0 = 1$, we infer that the plus-sign yields the x-coordinate of Q', while the minus-sign yields the x-coordinate of Q.
Putting it all together and also calculating the y-coordinates, we have shown the following *compound angle formulae*:

$$\boxed{\begin{aligned}
\sin(\theta+\phi) &= \sin\theta\,\cos\phi + \cos\theta\,\sin\phi \\
\cos(\theta+\phi) &= \cos\theta\,\cos\phi - \sin\theta\,\sin\phi \\
\sin(\theta-\phi) &= \sin\theta\,\cos\phi - \cos\theta\,\sin\phi \\
\cos(\theta-\phi) &= \cos\theta\,\cos\phi + \sin\theta\,\sin\phi
\end{aligned}}$$

You do not have to memorise these formulas, but you have to know that they exist and how to apply them. In particular, these formulas show that $\cos(60^\circ - 45^\circ) \neq \cos 60^\circ - \cos 45^\circ$!

Example: Find $\cos 15^\circ$ without using a calculator.

$$\begin{aligned}
\cos 15^\circ &= \cos(60^\circ - 45^\circ) \\
&= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ \\
&= \frac{1}{2} \times \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} \times \frac{1}{2}\sqrt{2} \\
&= \frac{1}{4}(\sqrt{2} + \sqrt{6}).
\end{aligned}$$

Using the compound angle formulas, one can deduce further relationships like, for example if $\theta = \phi$, the *double angle identities*:

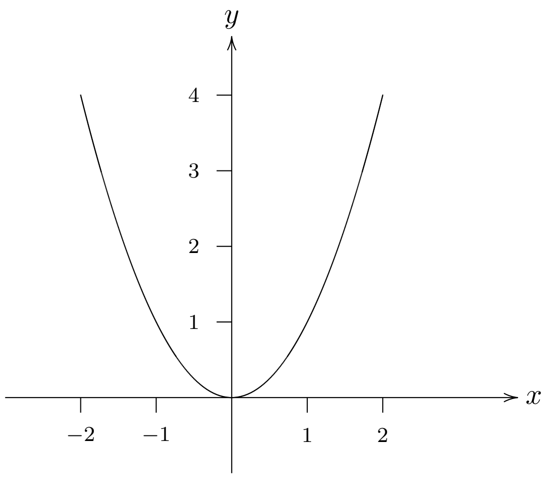
$$\sin(2\theta) = 2\sin\theta\cos\theta \quad \text{and} \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

§8 Graphs

This chapter is about sketching graphs to display their qualitative behaviour (without using graph paper).

A function f is a rule for calculating an output $f(x)$ from a given input x.
The equation $y = f(x)$ then gives a relation between y and x, which can be drawn on a graph.

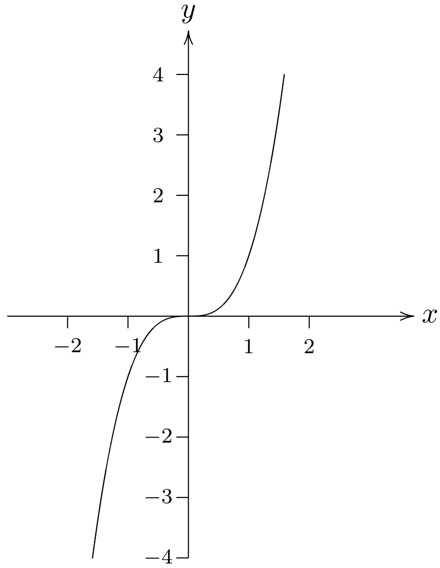
EXAMPLE: The graph of $y = x^2$. So, y is a function of x, i.e., $y = f(x)$ where $f(x) = x^2$.



NOTE: If $y^2 = x$, then y is not a function of x, because for $x > 0$ there are two possible values of y, namely $\sqrt{x}$ and $-\sqrt{x}$.

Odd and Even Functions
If $f(-x) = f(x)$ for all x, we call f *even*.
If $f(-x) = -f(x)$ for all x, we call f *odd*.

Examples and Remarks:



- $f(x) = x^2$ is even (see above).

  $f(x) = \cos(x)$ is even (see Lecture 10).

  The graph of an even function is symmetric about the y-axis, i.e., if (x, y) is on the graph, then so is $(-x, y)$.

- $f(x) = x^3$ is odd (see figure left).

  $f(x) = -\sin x$ is odd (see Lecture 10).

  The graph of an odd function is symmetric about the origin, i.e., if (x, y) is on the graph, so is $(-x, -y)$.

Quadratic Functions
These have the form: $f(x) = ax^2 + bx + c$.
We can employ *completing the square* to see how these functions behave.
Assuming $a \neq 0$, we can write:

$$\begin{aligned} f(x) = ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2 \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad (*) \end{aligned}$$

Since the square of any number is nonnegative, if $a > 0$ we can see directly from $(*)$ that $f(x) \geq c - b^2/4a$ and f achieves its least value at the point $x = -b/2a$.
Similarly, if $a < 0$, $f(x)$ takes its greatest value $c - \dfrac{b^2}{4a}$ at $x = -b/2a$.

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Example: Find the maximum value of $f(x) = -x^2-2x+2$ and find where it is achieved. Find the x- and y-intercepts and hence sketch the graph of $y = f(x)$.

Completing the square, we get

$$f(x) = -(x^2 + 2x) + 2 = -(x^2 + 2x + 1) + 3 = -(x + 1)^2 + 3.$$

Note that $(x + 1)^2 \geq 0$ for all x, wherefore $-(x + 1)^2 \leq 0$ for all values of x. Thus $f(x)$ cannot become greater than 3, and it achieves this value 3 if $(x + 1)^2 = 0$ (and this is the case for $x = -1$). So, the maximum value of $f(x)$ is 3 and this occurs at $x = -1$; the point $(-1, 3)$ lies on the curve.

The y-intercept occurs at $x = 0$ and is $y = f(0) = 2$, i.e., the point $(0, 2)$ lies on the curve.

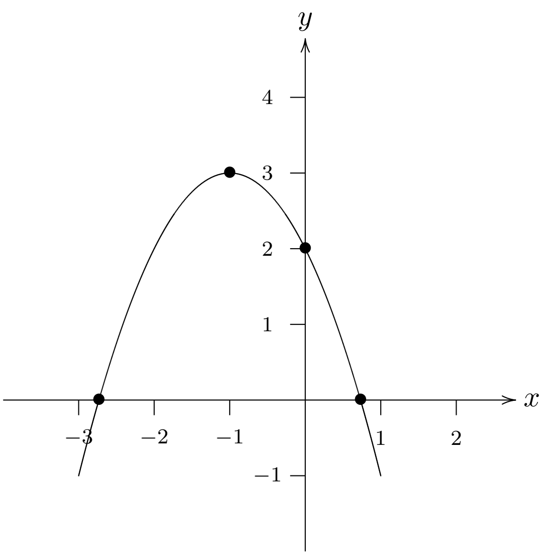
The x-intercepts occur when $y = 0$, i.e., $f(x) = 0$. Solving the quadratic equation $-(x + 1)^2 + 3 = 0$ we get $x = -1 \pm \sqrt{3}$, i.e., x-coordinates

$$x = -1 - \sqrt{3} = -2.732 \text{ (3 d.p.) and } x = -1 + \sqrt{3} = 0.732 \text{ (3 d.p.)}.$$

Thus we have two y-intercepts with coordinates $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$. (Note the symmetry: the x-coordinate of the maximum is exactly the mean value of the x-coordinates of the x-intercepts).

We sketch of the graph of $y = -x^2 - 2x + 2$:

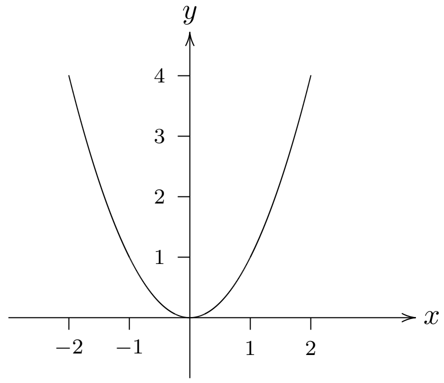
(we have calculated the points marked by •)



Behaviour of functions as $x \to \infty$ or $x \to -\infty$

The statement "$x \to \infty$" means that x gets arbitrarily large in the positive direction.
The statement "$x \to -\infty$" means that x gets arbitrarily large in the negative direction.

EXAMPLE: Consider the graph of $y = x^2$.



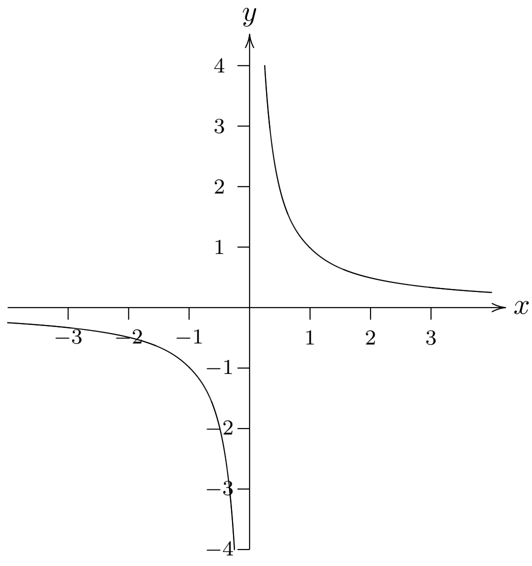
If x gets indefinitely large in the positive direction, then y is also positive and y increases indefinitely. Hence

$$y \to \infty \quad \text{as} \quad x \to \infty.$$

Also, if x is gets indefinitely large in the negative direction, then y is still positive and it still increases indefinitely. So

$$y \to \infty \quad \text{as} \quad x \to -\infty.$$

EXAMPLE: Consider the graph of $y = \frac{1}{x}$.



In this case, y approaches 0 from the positive side as $x \to \infty$, which we write as:

$$y \to 0+ \quad \text{as}\ x \to \infty.$$

Similarly

$$y \to 0- \quad \text{as} \quad x \to -\infty.$$

Also,

$$y \to +\infty \quad \text{as}\ x \to 0+$$

and

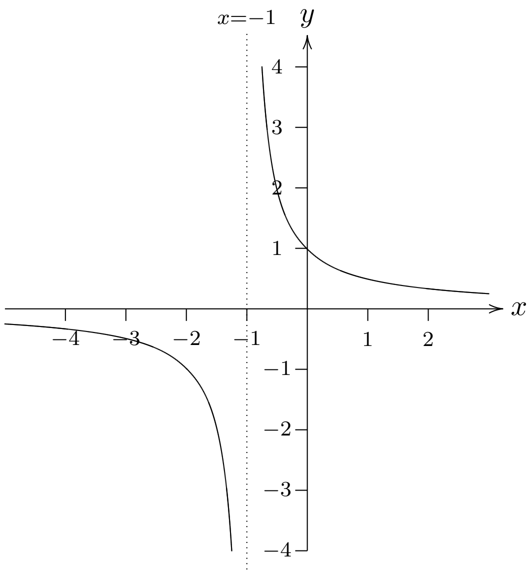
$$y \to -\infty \quad \text{as}\ x \to 0- \ .$$

Note that the bottom left hand part of the graph can be obtained from the top right by symmetry since the function $f(x) = \frac{1}{x}$ is odd – so the graph is symmetric about the origin.

An *asymptote* for a graph is a straight line which the graph gets arbitrarily close to. In the last example, i.e., the graph of $f(x) = \frac{1}{x}$, there are two asymptotes: $x = 0$ (the y-axis) and $y = 0$ (the x-axis).

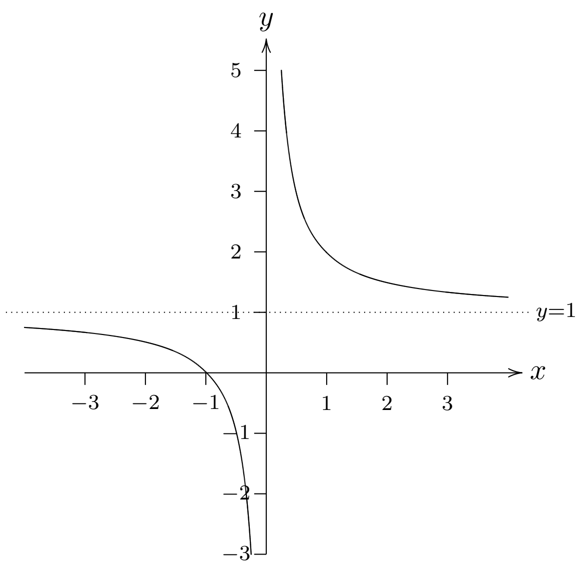
EXAMPLE: Consider the graph of $y = \dfrac{1}{x+1}$.

This has the same shape as $y = \dfrac{1}{x}$, but the asymptote $x = 0$ is shifted to $x = -1$. This is the dashed line in the picture left.



EXAMPLE: Consider the graph of $y = \dfrac{1}{x} + 1$.

This has the same shape as $y = \dfrac{1}{x}$, but the asymptote $y = 0$ is shifted to $y = 1$. This is the dashed line in the picture left.



Two obervations (simple transformations of curves)

- For any function f, the curve $y = f(x) + c$ is the translation of the curve $y = f(x)$ by c units parallel to the y-axis.
- For any function f, the curve $y = f(x + c)$ is the translation of the curve $y = f(x)$ by $-c$ units parallel to the y-axis.

Two further obervations on simple transformations of curves (not lectured)

- For any function f, the curve $y = a\,f(x)$ is the curve $y = f(x)$ stretched by a factor of a in y-direction.
  In particular, $y = -f(x)$ is the reflection of the curve $y = f(x)$ in the x-axis.
- For any function f, the curve $y = f(a\,x)$ is the curve $y = f(x)$ stretched by a factor of $\frac{1}{a}$ in x-direction.
  In particular, $y = f(-x)$ is the reflection of the curve $y = f(x)$ in the y-axis.

EXAMPLE: The graph of

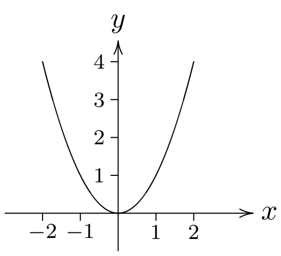
$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

is obtained from $f(x) = x^2$ by first stretching by a factor of a in y-direction, and then translating by $-\frac{b}{2a}$ parallel to the x-axis and by $c - \frac{b^2}{4a}$ parallel to the y-axis.
So, for the example at the beginning of this lecture, namely

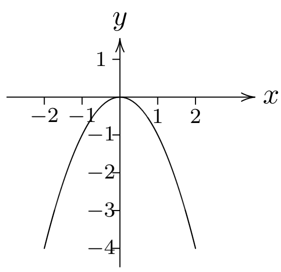
$$f(x) = -x^2 - 2x + 2 = -(x+1)^2 + 3,$$

we have:

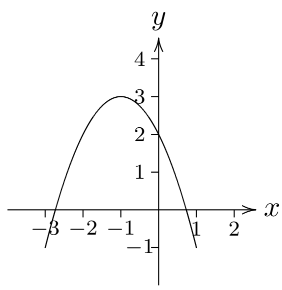
the graph of $f(x) = x^2$:



reflected in x-axis:



translating -1 in x and 3 in y:



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