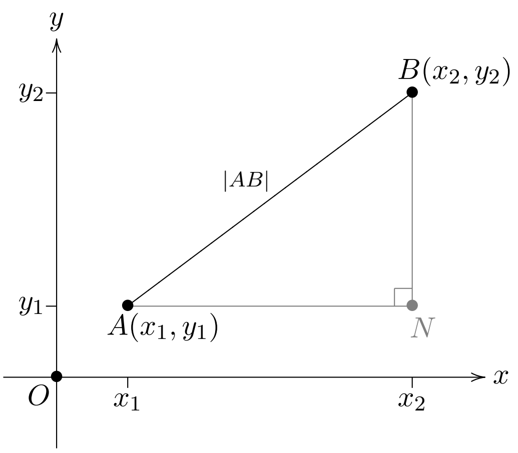
MA10103: Foundation Mathematics I

Lecture Notes – Week 7

§7. Coordinate Geometry

Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$



Apply Pythagoras' Theorem to triangle ANB (note that N has coordinates (x_2, y_1)):

$$\begin{aligned} |AB|^2 &= |AN|^2 + |NB|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$

Formula is:

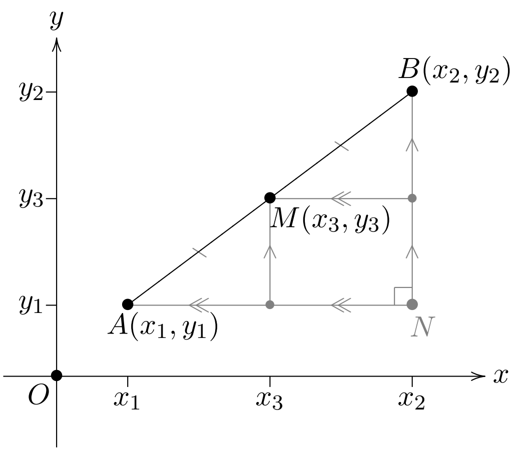
$$\boxed{|AB| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

("Square root of the sum of the difference in the x-coordinates squared and the y-coordinates squared")

Examples: Find distances between the following pairs of points:

- $A(1, 3)$ and $B(4, 5)$: $|AB| = \sqrt{(4-1)^2 + (5-3)^2} = \sqrt{13}$.
- $A(1, 3)$ and $B(4, -1)$: $|AB| = \sqrt{9 + 16} = 5$.
- $A(1, 3)$ and $B(1, 6)$: $|AB| = \sqrt{9} = 3$.

Midpoint between A and B



If the lengths of the lines marked by "$\setminus$" are the same, so are the ones marked by "$\ll$" and by "$\wedge$", respectively.
So, for the differences we have

$$x_3 - x_1 = \frac{1}{2}(x_2 - x_1) \quad \text{and}$$
$$y_3 - y_1 = \frac{1}{2}(y_2 - y_1).$$

From this, we get for the coordinates (x_3, y_3) for the midpoint M:

$$\boxed{x_3 = \tfrac{1}{2}(x_1 + x_2) \ \text{ and } \ y_3 = \tfrac{1}{2}(y_1 + y_2)}$$

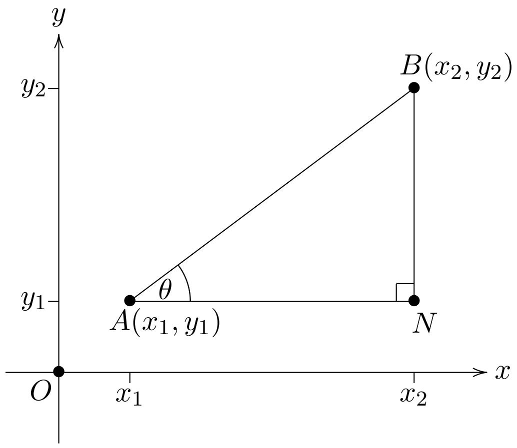
("Mean of the coordinates")

Examples: Find the midpoint between the following pairs of points:

- $A(3,-1)$ and $B(2,7)$: midpoint has coordinates $(5/2,3)$.
- $A(0,5)$ and $B(0,7)$: midpoint has coordinates $(0,6)$.

Gradient of a line

To specify a line, it is enough to give one point on it and the *gradient* of the line. The gradient is defined as "(increase in y) / (increase in x)" between any two points on the line.



For the line AB:

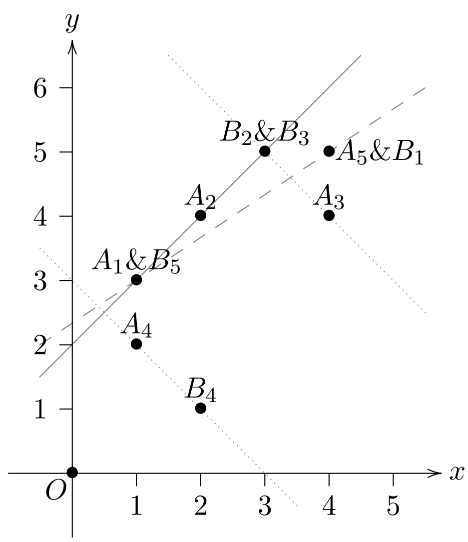
$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

In fact, Gradient $= \tan\theta$ (where θ is the angle in the picture left).

Example: The slope of a road is given as value of the gradient. E.g., according to its road signs, Bathwick Hill has a slope of 11%, so its gradient is 0.11 (this means that for every 100 metres (or yards, if you prefer) in horizontal direction it goes up 11 metres (or yards, respectively). The corresponding angle is $\tan 0.11 = 6.3°$ (1 d.p.).

Further Examples: Calculate the gradient of the line through

- $A_1(1,3)$ and $B_1(4,5)$: $\frac{2}{3}$
- $A_2(2,4)$ and $B_2(3,5)$ 1
- $A_3(4,4)$ and $B_3(3,5)$ -1
- $A_4(1,2)$ and $B_4(2,1)$ -1
- $A_5(4,5)$ and $B_5(1,3)$ $\frac{2}{3}$

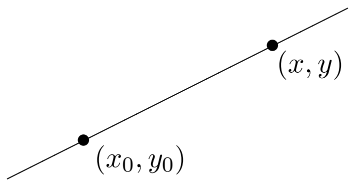


Properties of Straight Lines

- Parallel lines have the same gradient.
- Lines parallel to the x axis have zero gradient.
- Lines parallel to the y axis have an infinite gradient (the undefined division by 0).
- Positive gradient - slopes upwards. Negative gradient - slopes downwards.

Equation of a line
Suppose a point (x_0, y_0) on the line and the gradient m of the line are given. Then any other point (x, y) on the line satisfies

$$\frac{y - y_0}{x - x_0} = m.$$



This can be written as

$$\boxed{y - y_0 = m(x - x_0)} \qquad \text{respectively} \qquad \boxed{y = mx + (y_0 - x_0)}$$

This is (or these are, if you like) the equation of the line.

NOTE: The above equations exclude lines parallel to the y-axis (i.e., lines with infinite slope). Such a line parallel to the y-axis through the point $(x_0, 0)$ (or any point (x_0, y_0)) has the equation $x = x_0$.

EXAMPLES:

(a) Find the equation of the line through $(1, 1)$ with gradient 3.
$y - 1 = 3(x - 1)$ or $y = 3x - 2$.

(b) Find the equation of the line through $(1, 1)$ and $(3, 4)$.
The gradient is $\frac{4-1}{3-1} = \frac{3}{2}$. Therefore, the line has the equation $y = \frac{3}{2}x + c$ and we have to find c. Since the line goes through $(1, 1)$, we have $1 = \frac{3}{2} \times 1 + c$ (or, $4 = \frac{3}{2} \times 3 + c$) and hence $c = -\frac{1}{2}$. So, the equation of this line is $y = \frac{3}{2}x - \frac{1}{2}$.

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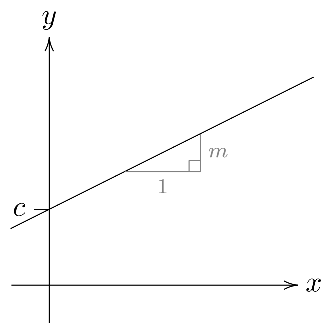
The equation of a line can be rewritten as

$$\boxed{y = mx + c}$$

with $c = y_0 - mx_0$.

c is the *y-intercept* of the line (where it crosses the y axis).



EXAMPLE: A line has equation $2x + 3y = 6$. Find its gradient, and its $y-$ intercept. Since $y = -\frac{2}{3}x + 2$, the gradient is $-\frac{2}{3}$ and the y-intercept is 2.

Intersection of two lines

Given two lines, the intersection point(s) (if any) are found by solving the equations of the lines simultanesouly.

EXAMPLES:

(a) Find the intersection point of

$$y = 2x + 3 \quad \text{and} \quad y = x + 2.$$

Subtract to get $x + 1 = 0$, which gives $x = -1$. Sustitute this into second equation to get $y = 1$. So, the point of intersection has coordinates $(-1, 1)$ (check that $(-1, 1)$ satisfies both equations).

(b) Find the point of intersection of the two lines given by $2x+3y = 7$ and $3x-2y = 4$. Solve the two equations simultaneously to get the intersection point $(2, 1)$ (e.g., by substracting 3 times the first from twice the second equation; this yields $13y = 13$).

Remark (not lectured)

Given two line ℓ_1, ℓ_2 in the plane, there are 3 possibilities:

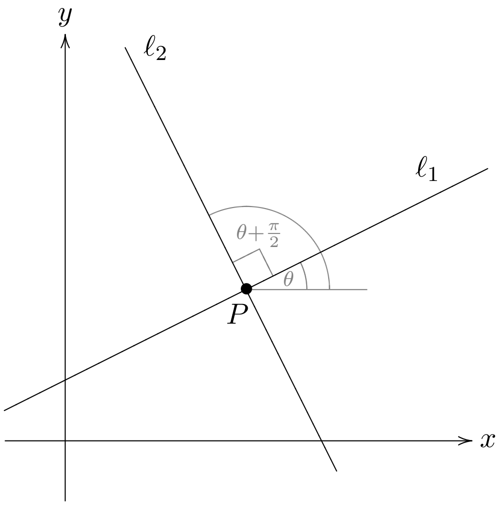
(i) ℓ_1 and ℓ_2 are non-parallel – they intersect in a single point.

(ii) ℓ_1 and ℓ_2 are parallel and distinct – they do not intersect.

(iii) $\ell_1 = \ell_2$ – so their intersection has infinitely many points.

Perpendicular lines

Suppose lines ℓ_1, ℓ_2 with gradients m_1, m_2 respectively intersect at right angle at P



What is the relation between m_1 and m_2?

The gradient m_1 (of ℓ_1) is given by $m_1 = \tan\theta$ (see figure on the left). The gradient m_2 (of ℓ_2) is given by

$$m_2 = \tan\left(\theta + \frac{\pi}{2}\right) \overset{\text{Lecture 11}}{=} -\frac{1}{\tan\theta} = -\frac{1}{m_1}.$$

So, for the gradients m_1 and m_2 of perpendicular lines we have

$$\boxed{m_1 \times m_2 = -1}$$

If we know either m_1 or m_2, we can find the other.

EXAMPLES

(a) What is the gradient of lines perpendicular to $y = 2x + 4$?
The gradient is $-1/2$.

(b) Find the equation of the line through $(1, 1)$ perpendicular to $y = 2x + 4$.
This line is given by $y - 1 = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x + \frac{3}{2}$.

(c) Find the equation of the line perpendicular to $y = 2x + 4$ and having the same y-intercept?
We immediately get $y = -\frac{1}{2}x + 4$.

Equation of a circle

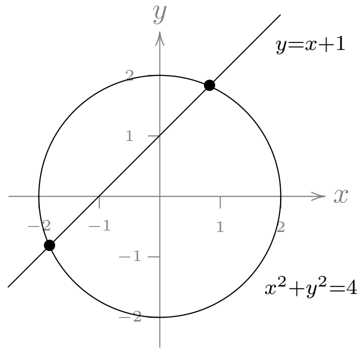
The circle with centre $P(x_0, y_0)$ and radius r comprises all points $Q(x, y)$ satisfying $|PQ| = r$, i.e.,

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$

Squaring this equation, the circle around (x_0, y_0) of radius r has the equation:

$$\boxed{(x - x_0)^2 + (y - y_0)^2 = r^2}$$

EXAMPLE: Find the point(s) of intersection of the line $y = x+1$ and the circle $x^2+y^2 = 4$ (the circle of radius 2 around the origin).



Substitute $y = x + 1$ in the circle to get: $x^2 + (x + 1)^2 = 4$. Simplifying this we get $2x^2 + 2x - 3 = 0$. Using the formula for solutions of a quadratic equation, gives

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 + 4 \times 3 \times 2}}{4} \\ &= \frac{-2 \pm 2\sqrt{7}}{4} \\ &= -\frac{1}{2} \pm \frac{\sqrt{7}}{2}. \end{aligned}$$

Substitute each of these values of x into the equation of the straight line to find y.
The points of intersection are $(-\frac{1}{2} + \frac{\sqrt{7}}{2}, \frac{1}{2} + \frac{\sqrt{7}}{2})$ and $(-\frac{1}{2} - \frac{\sqrt{7}}{2}, \frac{1}{2} - \frac{\sqrt{7}}{2})$ (see picture).

END OF LECTURE 14