MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 6

Last week, we explored $\sin\theta$ and $\cos\theta$ in a quite general way. We found that it is actually enough to know the values of these functions in the range $0 \leq \theta \leq \frac{\pi}{2}$, e.g., for $\sin\theta$ we found

$$\sin(\theta) = \sin(\theta + 2\pi) \overset{\text{similarly}}{=} \sin(\theta + 4\pi) = \sin(\theta + 6\pi) = \ldots = \sin(\theta - 2\pi) = \sin(\theta - 4\pi),$$

i.e., sine is 2π-periodic (so if we know sine for angles $0 \leq \theta \leq 2\pi$, we know it everywhere), and

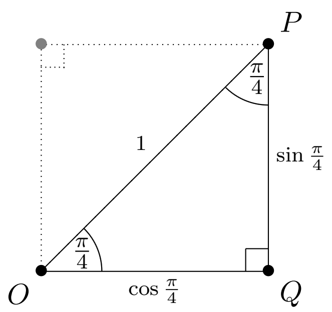
$$\begin{aligned}
\sin(-\theta) &= -\sin(\theta) && \text{(yields values for } \tfrac{3\pi}{2} \leq \theta \leq 2\pi\text{, if the ones for } 0 \leq \theta \leq \tfrac{\pi}{2} \text{ are known)}\\
\sin(\pi - \theta) &= \sin(\theta) && \text{(yields values for } \tfrac{\pi}{2} \leq \theta \leq \pi\text{, if the ones for } 0 \leq \theta \leq \tfrac{\pi}{2} \text{ are known)}\\
\sin(\pi + \theta) &= -\sin(\theta) && \text{(yields values for } \pi \leq \theta \leq \tfrac{3\pi}{2}\text{, if the ones for } 0 \leq \theta \leq \tfrac{\pi}{2} \text{ are known).}
\end{aligned}$$

Moreover, since

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{and} \quad \tan\theta = \frac{\sin\theta}{\cos\theta},$$

it is enough to know to know one of the trigonometric functions $\sin\theta$, $\cos\theta$ or $\tan\theta$ in the range $0 \leq \theta \leq \frac{\pi}{2}$ to know all of them everywhere!

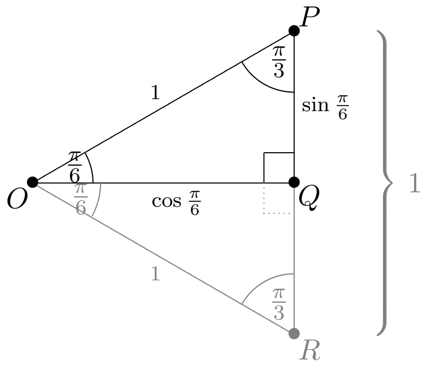
Special values of $\sin\theta$ and $\cos\theta$



We determine $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$ (note that $\frac{\pi}{4}$ radians is 45°):
Since the angles of the triangle add up to π (or 180°), we have for the triangle OPQ the situation on the left (note that this is half of the square with diagonal of length 1). Thus we have that the lengths $|OQ| = |QP|$, i.e., $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$, and Pythagoras' theorem yields $1 = 2\sin^2\frac{\pi}{4}$ and hence

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}.$$

This also yields $\tan\frac{\pi}{4} = 1$.



We determine $\sin\frac{\pi}{6}$ and $\cos\frac{\pi}{6}$ ($\frac{\pi}{6}$ radians is 30°): We can extend the triangle OQP to the triangle ORP. The latter one is a equilateral triangle with all sidelengths equal to 1 ("equilateral" means "equal sides"). So we can read off

$$\sin\frac{\pi}{6} = |PQ| = \frac{1}{2}$$

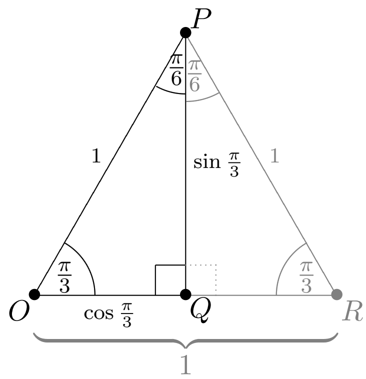
and Pythagoras yields

$$\cos^2\frac{\pi}{6} = 1 - \sin^2\frac{\pi}{6} = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4},$$

$$\text{hence } \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

So, we also have

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$



We determine $\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{3}$ ($\frac{\pi}{3}$ radians is 60°): We exchange the role of P and O in the last picture above and obtain:

$$\cos\frac{\pi}{3} = \frac{1}{2},$$
$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$
$$\tan\frac{\pi}{3} = \sqrt{3}.$$

Actually, the argument we used here, holds more generally and reads

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \quad \text{and} \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

Consequently, we also have

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}.$$

This even means that it is enough to know one of the trigonemetric functions in the range $0 \le \theta \le \frac{\pi}{4}$.

This last rule connects sine and cosine at different(!) angles. Using $\sin(\pi - \theta) = \sin\theta$, we also have

$$\sin(\pi - \theta) = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right).$$

Using the substitution $\phi = \frac{\pi}{2} - \theta$ on the left and the right hand side, one obtains

$$\sin\left(\phi + \frac{\pi}{2}\right) = \cos\phi. \tag{1}$$

Similarly, one also obtains

$$\cos\left(\phi + \frac{\pi}{2}\right) = -\sin\phi \quad \text{and also} \quad \tan\left(\phi + \frac{\pi}{2}\right) = -\frac{1}{\tan\phi}.$$

Equation (1) means that the value of cosine at an angle ϕ is the same as the value of sine at the angle $\phi + \frac{\pi}{2}$ ("ϕ shifted by $\frac{\pi}{2}$").

We summarise our findings of special values in a table:

θ in radians	θ in degrees	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	$0°$	0	1	0
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	$90°$	1	0	not defined
$\frac{2\pi}{3}$	$120°$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	$135°$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$\frac{5\pi}{6}$	$150°$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
π	$180°$	0	-1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

For angles greater than $\frac{\pi}{2}$ use either the shift formula or the formulas obtained in the last lecture.

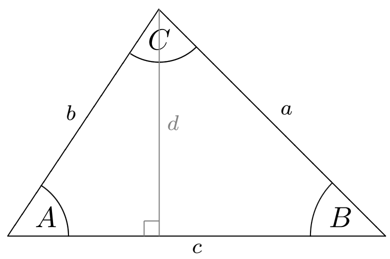
Note: The value of $\tan\theta$ becomes arbitrary large if we approach the angle $\theta = \frac{\pi}{2}$ "from below" (you may use your calculator to calculate $\tan(\frac{\pi}{2} - 0.1)$, $\tan(\frac{\pi}{2} - 0.01)$, $\tan(\frac{\pi}{2} - 0.001)$ etc.). However, If we approach $\theta = \frac{\pi}{2}$ "from above" (e.g., $\tan(\frac{\pi}{2} + 0.1)$, $\tan(\frac{\pi}{2} + 0.01)$, $\tan(\frac{\pi}{2} + 0.001)$ etc.), we get arbitrary large negative values.

Example: For what values of θ between -2π and 2π is $\cos\theta = \frac{\sqrt{3}}{2}$?
One solution is $\theta = \frac{\pi}{6}$. Since $\cos(-\theta) = \cos(\theta)$, the angle $\theta = -\frac{\pi}{6}$ is also a solution. Rotating the first through -2π and the second through 2π (2π-periodicity!), we also get $\theta = -\frac{11\pi}{6}$ and $\theta = \frac{11\pi}{6}$ (i.e, $\pm 330°$).
So, solutions are $\theta = \pm\frac{\pi}{6}$ and $\pm\frac{11\pi}{6}$. These are all the solutions in the specified range.

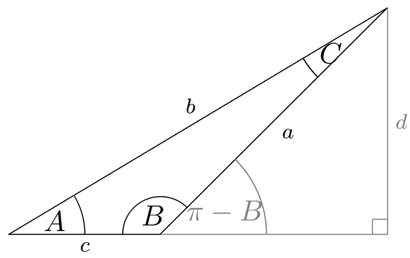
Finally, we now return to triangles. What are the relationsships between angles and sides in general (i.e., not right angled) triangles? We derive such relationships using our knowledge of right angled triangles.

The sine rule
Label the edge lengths and angles of any triangle as shown:

If all angles are acute (less than $\frac{\pi}{2}$):



If one angle is obtuse (more than $\frac{\pi}{2}$):



We know how to calculate the length of the height d (using our knowledge of right angled triangles):

- Using b as hypotenuse and the angle A, we obtain the opposite d as $d = b \sin A$.

- We can also use, in the acute case, a as hypotenuse and the angle B to obtain $d = a \sin B$.
  In the obtuse case, we have to use the angle $\pi - B$ to get $d = a \sin(\pi - B) = a \sin B$, since $\sin\theta = \sin(\pi - \theta)$ for any angle θ.

Equating these two equations for d we have $b \sin A = a \sin B$, or easier to memorise (and with similar reasoning applied to the remaining two heights)

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

This is the *sine rule* (note that the side a is opposite to the angle A etc.).

END OF LECTURE 11

In proving the sine rule, we determined the height d perpendicular to the side c. But with this knowledge, we can also calculate the area of a triangle.

The area of a triangle

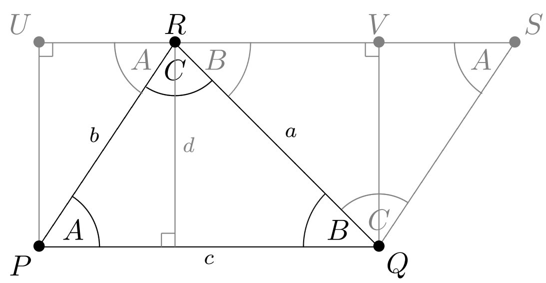
The area of a triangle is given by: $\boxed{\frac{1}{2} \times \text{side} \times \text{perpendicular height}}$

Hence, the area of the triangle is $\frac{1}{2}cd$ or, using the above expressions for d,

$$\text{area of an triangle} = \frac{1}{2}bc\,\sin A = \frac{1}{2}ac\,\sin B = \frac{1}{2}ab\,\sin C.$$

(half of the product of two of its side lengths times the sine of the enclosed angle)

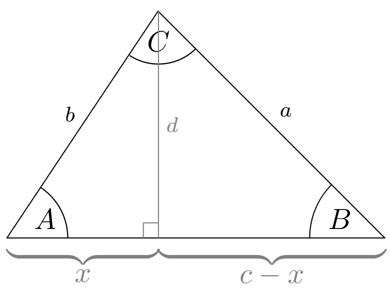
Proof of the area formula for a triangle (not lectured, not examinable)



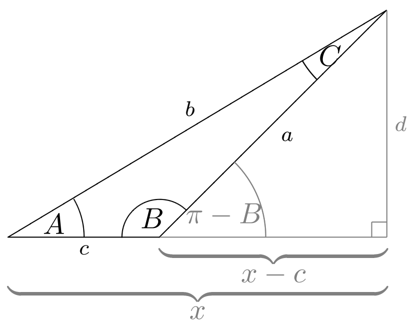
The formula for the area of a triangle can be deduced from the picture on the left: The area of the triangle PQR is twice the area of the parallelogram $PQSR$. But this parallelogram $PQSR$ has the same area as the rectangle $PQUV$, since the triangles QSV and PRU have the same areas. Thus, the area of this rectangle is given by the product of the lengths of the sides PQ and QS, i.e., by cd.

The cosine rule

If all angles are acute (less than $\frac{\pi}{2}$):



If one angle is obtuse (more than $\frac{\pi}{2}$):



Suppose, we know the lengths x or $c - x$. Then we can calculate d using Pythagoras' theorem:

$$d^2 = b^2 - x^2 \quad \text{or} \quad d^2 = a^2 - (c-x)^2 = a^2 - (x-c)^2 = a^2 - x^2 - c^2 + 2xc.$$

Combining these two equations yields an equation for x, namely

$$b^2 \cancel{-x^2} = a^2 \cancel{-x^2} - c^2 + 2xc,$$

hence

$$x = \frac{b^2 + c^2 - a^2}{2c}.$$

We also have $x = b \cos A$ and therefore

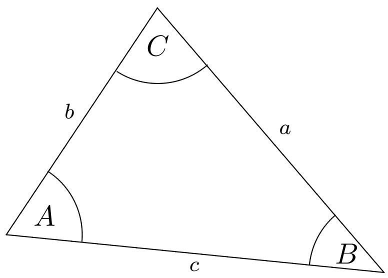
$$\boxed{\cos A = \frac{b^2 + c^2 - a^2}{2bc}} \qquad \text{or} \qquad \boxed{a^2 = b^2 + c^2 - 2bc \cos A}$$

This is the *cosine rule* and generalises Pythagoras' theorem to general triangles. Of course, the formulas

$$b^2 = a^2 + c^2 - 2ac \cos B \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab \cos C$$

also hold.

EXAMPLES: We always have the following situation, i.e., A is the angle opposing the side a etc.



1.) Given $a = 8$, $b = 10$ and $c = 3$, calculate angles.
Use cosine rule to calcculate A:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{100 + 9 - 64}{2 \times 10 \times 3} = \frac{45}{60} = \frac{3}{4}.$$

Hence, (using a calculator) one finds $A = 41.41°$ (2 d.p.).

(Note: from this we also have $\sin^2 A = 1 - \cos^2 A = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}$ and therefore $\sin A = \frac{\sqrt{7}}{4}$, where $\sin A$ is positive since the angle A is between $0°$ and $180°$).

One can now continue either with the cosine or the sine rule.

a) **Using the cosine rule** (more complicated):

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{64 + 9 - 100}{48} = -\frac{9}{16},$$

hence $B = 124.23°$ (2 d.p.).
Also, we then have $C = 180° - A - B = 14.36°$ (2 d.p.).

b.) **Using the sine rule:**

$$\sin B = \frac{b}{a}\sin A = \frac{10}{8} \times \frac{\sqrt{7}}{4} = \frac{5\sqrt{7}}{16},$$

and hence $B = 55.77°$ (2 d.p.) or $B = 180° - 55.77° = 124.23°$ (2 d.p.) (unfortunately, we cannot excluded one of the two possibilities immediately, so we also have to calculate C using sine rule).

$$\sin C = \frac{c}{a}\sin A = \frac{3}{8} \times \frac{\sqrt{7}}{4} = \frac{3\sqrt{7}}{32},$$

hence $C = 14.36°$ (2 d.p.) or $C = 180° - 14.36° = 165.64°$ (2 d.p.). The second possibility can be excluded since $35° + 165.64° > 180°$, so $C = 14.36°$ (2 d.p.) and consequently also $B = 124.23°$ (2 d.p.).

2.) Given $B = 35°$, $b = 5$ and $c = 7$, calculate all the remaining side lengths and angles.
Using the sine rule we have

$$\sin C = \frac{c}{b}\sin B = \frac{7}{5}\sin 35°,$$

hence $C_1 = 53.42°$ (2 d.p.) or $C_2 = 180° - 53.42° = 126.58°$ (2 d.p.).

Continuing with these values, we also have

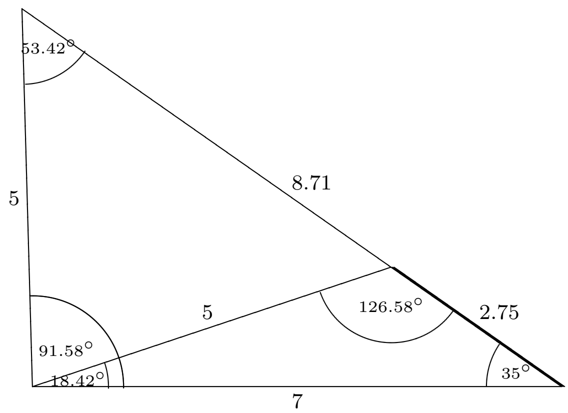
$A_1 = 180° - B - C_1 = 91.58°$ (2 d.p.) and $A_2 = 180° - B - C_2 = 18.42°$ (2 d.p.).

Using the sine rule again, we obtain

$$a_1 = b\frac{\sin A_1}{\sin B} = 5\frac{\sin A_1}{\sin 35°} = 8.71 \text{ (2 d.p.)} \quad \text{and}$$

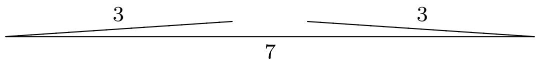
$$a_2 = b\frac{\sin A_2}{\sin B} = 5\frac{\sin A_2}{\sin 35°} = 2.75 \text{ (2 d.p.)}.$$

Note, we actually have two solutions here:



Remarks and Further Examples (not lectured)

Remark: Not all choices of values for the side lengths a, b and c yield a triangle, the sum of the two shorter sides must be greater than the larger side (then, of course, the sum of any two sides is larger than the third side):



This is implicit in the cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$, since $-1 \leq \cos A \leq 1$, so we have

$$(b-c)^2 = (c-b)^2 = b^2 + c^2 - 2bc \leq b^2 + c^2 - 2bc \cos A \leq b^2 + c^2 + 2bc = (b+c)^2,$$

and therefore

$$(b-c)^2 = (c-b)^2 \ \leq \ a^2 \ \leq \ (b+c)^2.$$

Taking the square roots, we get the inequalities $a \leq b+c$, $b \leq a+c$ and $c \leq a+b$. These inequalities are called *triangle inequality*.

Example: Given $B = 35°$, $b = 7$ and $c = 5$ (compare this with the last example), calculate all the remaining side lengths and angles.
Using the sine rule we have

$$\sin C = \frac{c}{b} \sin B = \frac{5}{7} \sin 35°,$$

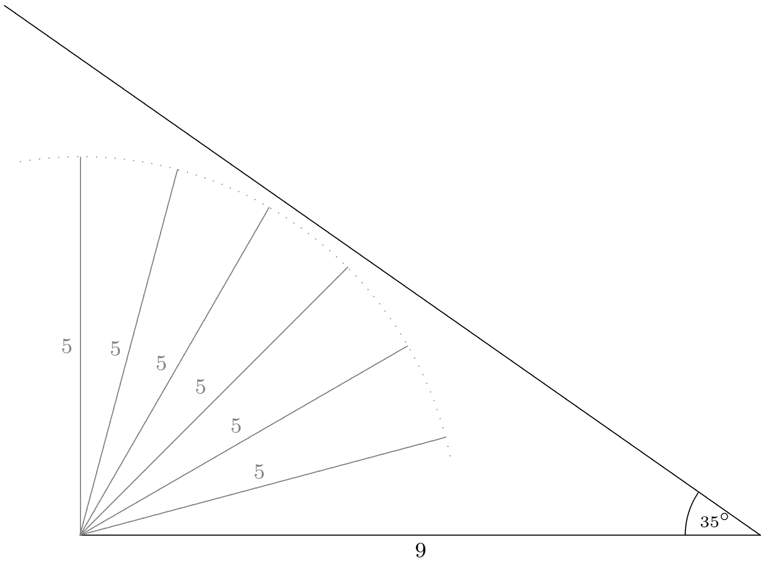
hence $C = 24.19.°$ (2 d.p.) (two other value $180° - 24.19° = 155.81°$ (2 d.p.) is not possible, since the sum of the angles has to be less than or equal to $180°$).
So, we also obtain $A = 180° - B - C = 120.82°$ (2 d.p.). Using the sine rule again , we calculate

$$a = b\,\frac{\sin A}{\sin B} = 7\,\frac{\sin A}{\sin 35°} = 10.48 \text{ (2 d.p.)}.$$

Example and Remark: Given $B = 35°$, $b = 5$ and $c = 9$ (compare this with the last examples), calculate all the remaining side lengths and angles.
Using the sine rule we have

$$\sin C = \frac{c}{b} \sin B = \frac{9}{5} \sin 35° = 1.032\ldots$$

Since $sinC$ cannot be greater than 1, we do not get a solution for C here. In fact, the given values for b, c and B cannot belong to a triangle (the side $b = 5$ is too short to intersect the line defined by the angle $B = 35°$):



END OF LECTURE 12