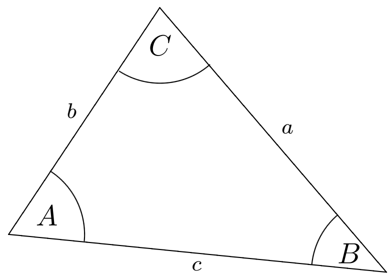
MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 5

§6 TRIGONOMETRY

Motivation:
Consider the triangle:



For any triangle, the sum of its interior angles is 180°, i.e., $A + B + C = 180°$.
If we know two angles we can calculate the third. For example, if $A = 30°$ and $B = 70°$, then $C = 180° - 100° = 80°$.

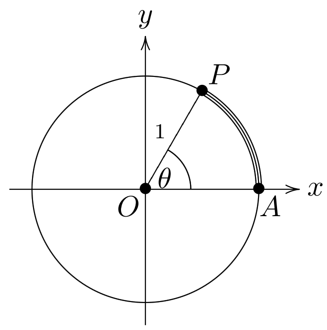
But if we know two sides and one angle, or one side and 2 angles etc., can we calculate the remaining sides and angles? And if yes, how can we calculate them?
These are the guiding questions in this chapter!

Measuring angles
There are two units for measuring angles, *degrees* and *radians*.
A full circle is 360° (a rather arbitrary number/unit introduced by the Babylonians) and is also 2π radians. To convert degrees to radians we simply multiply by $\frac{\pi}{180}$:

$$\boxed{\text{angle in degrees} \times \frac{\pi}{180} \ = \ \text{angle in radians}}$$

degrees	0°	30°	45°	60°	90°	180°	270°	360°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Why do we introduce radians? Radians is a "natural" measure measure for an angle: Take a circle of radius 1 (the *unit circle*). Consider the segment AP of the circumference formed by the angle θ radians. Note that the circumference of the unit circle is 2π.



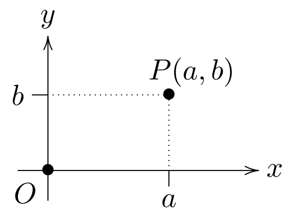
If this segment is all of the unit circle, then $\theta = 2\pi$ and the length of the segment is 2π.
If this segment is half of the unit circle, then $\theta = \pi$ and the length of the segment $\frac{1}{2} \times 2\pi = \pi$.
And so on...

The general formula is

$$\boxed{\text{length of segment of unit circle formed by angle } \theta \text{ radians} \;=\; \theta \text{ units}}$$

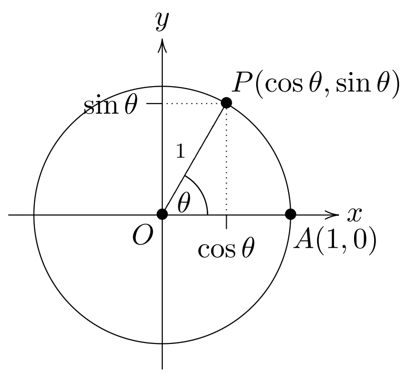
This formula is simple **provided** θ is measured in radians.

A word on coordinate geometry (more to come in the next chapter)
Consider the plane with x and y axes and origin O (with coordinates $(0, 0)$). Any point P can be identified by its x- and y-coordinate.



Definition of $\sin\theta$ **and** $\cos\theta$

Consider again the unit circle. Let P be the point you get to by rotating around the unit circle through an angle θ in the anticlockwise direction from $A(1, 0)$. The point P, of course, depends on the angle θ.



$\cos\theta$ is *defined* to be the x coordinate of P.
$\sin\theta$ is *defined* to be the y coordinate of P.

So, P has coordinates $(\cos\theta, \sin\theta)$.

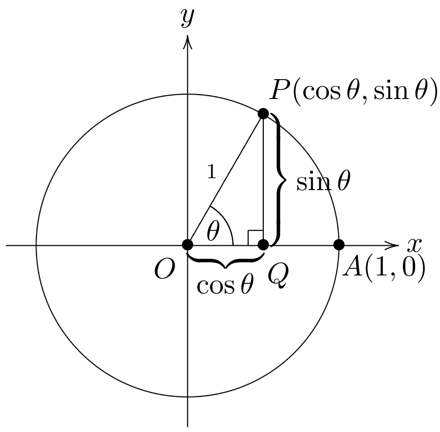
It also follows that

$$\cos 0 = 1 = \sin\frac{\pi}{2} \qquad\qquad \sin 0 = 0 = \cos\frac{\pi}{2}.$$

Applying Pythagoras' theorem in the right angled triangle OQP, we get

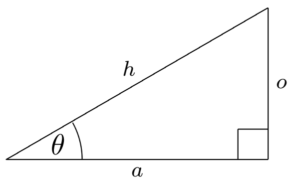
$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

Note: $\cos^2\theta$ means $(\cos\theta)^2$ (and **not** $\cos(\cos\theta)$).

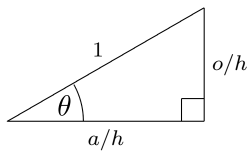


Equivalent way to compute $\sin\theta$ and $\cos\theta$ (for angles $0 \le \theta \le \pi/2$)

Consider any right angled triangle. Here, h is short for *hypotenuse*, a for *adjacent* (the side adjacent to the angle θ) and o for *opoosite* (the side opposite of the angle θ).



If you scale all sides by a factor of $1/h$, all angle stay the same, and the hypotenuse of this triangle has now length 1. Comparing this situation with the definition of cos and sin above, we have:

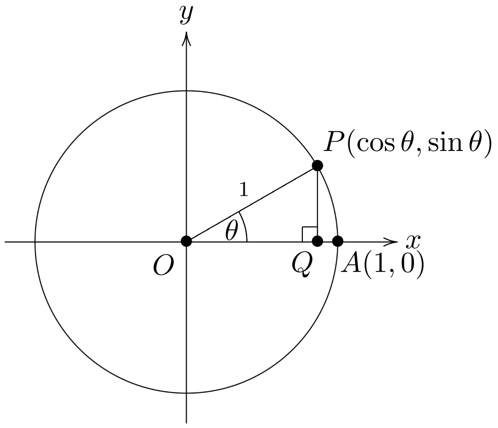


$$\cos\theta = \frac{a}{h} = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\sin\theta = \frac{o}{h} = \frac{\text{opposite}}{\text{hypotenuse}}$$

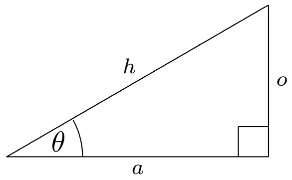
WARNING: This holds **only** in right angled triangles!

END OF LECTURE 9

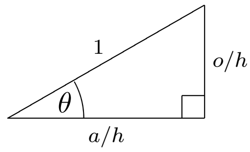
Recall: We have



and



or, after scaling with $\frac{1}{h}$,



Comparing these right angled triangles, we have

$$\cos\theta = \tfrac{a}{h} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sin\theta = \tfrac{o}{h} = \frac{\text{opposite}}{\text{hypotenuse}}$$

The definition of $\tan\theta$ is

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

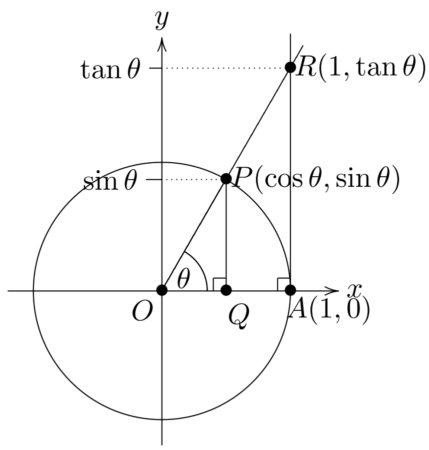
In a right angled triangle, this is the same as

$$\tan\theta = \frac{o}{a} = \frac{\text{opposite}}{\text{adjacent}}.$$

Geometric interpretation of the tangent function & definition of $\cot\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ (not examinable)

The length $\tan\theta$ can be found as length of the line AR in the picture on the right (i.e., as y-coordinate of the point of intersection of the vertical tangent on the unit circle and extended line through OP).

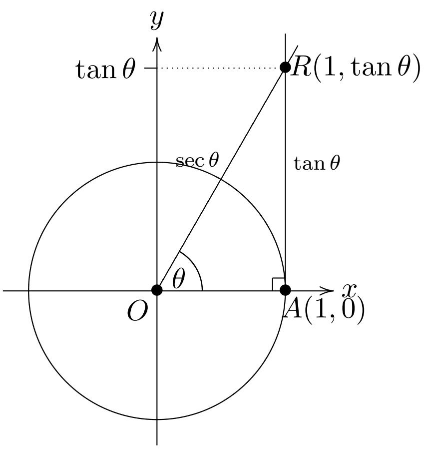
So, one gets the value of $\tan\theta$ by looking at the (vertical) **tan**gent!

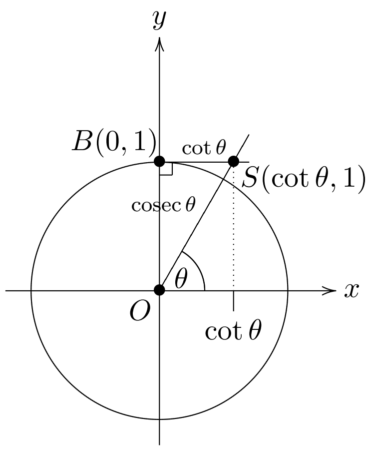


In a right angled triangle, $\cos\theta$, $\sin\theta$ and $\tan\theta$ are given as ratio of two of its sides. There are three more such ratios and they all have their own name, namely *cotangent*, *secant* and *cosecant*. However, they are just the inverse values of the previous ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$:

$$\begin{aligned}\cot\theta &= \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan\theta}\\ \sec\theta &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos\theta}\\ \operatorname{cosec}\theta &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin\theta}\end{aligned}$$

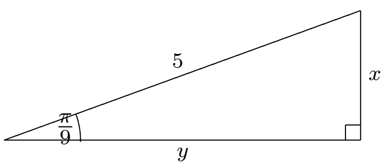
But geometrically, they correspond to the lengths of the sides of the triangle defined via the vertical respectively horizontal tangent on the unit circle.





Simple uses of trigonometry (right angled triangles)
If we know an angle θ (besides the right angle) and one side we can find other sides of a triangle.

Example: Find the lengths of the sides x and y.



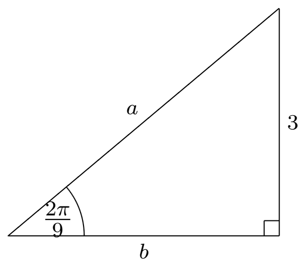
To find x: $\sin\frac{\pi}{9} = \frac{x}{5}$.
Hence $x = 5\,\sin\frac{\pi}{9} = 1.71$ (2 d.p.).

To find y: $\cos\frac{\pi}{9} = \frac{y}{5}$.
Hence $y = 5\,\cos\frac{\pi}{9} = 4.70$ (2 d.p.).

Note that $\frac{\pi}{9}$ radians is $20°$.

Further Example (not lectured)

Find the lengths of the sides a and b.

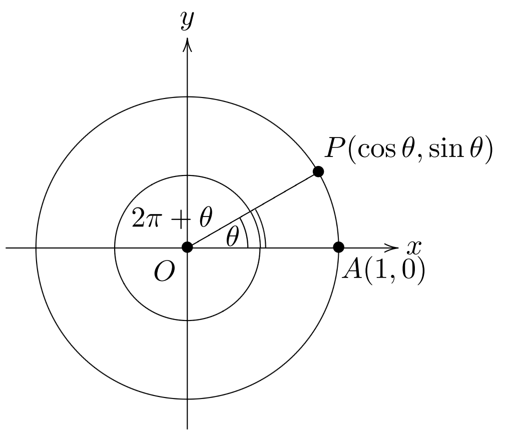


To find a: $\sin\frac{2\pi}{9} = \frac{3}{a}$.
Hence $a = \frac{3}{\sin\frac{2\pi}{9}} = 4.67$ (2 d.p.).

To find y: $\tan\frac{2\pi}{9} = \frac{3}{b}$.
Hence $b = \frac{3}{\tan\frac{2\pi}{9}} = 3.58$ (2 d.p.).

Note that $\frac{2\pi}{9}$ radians is $40°$.

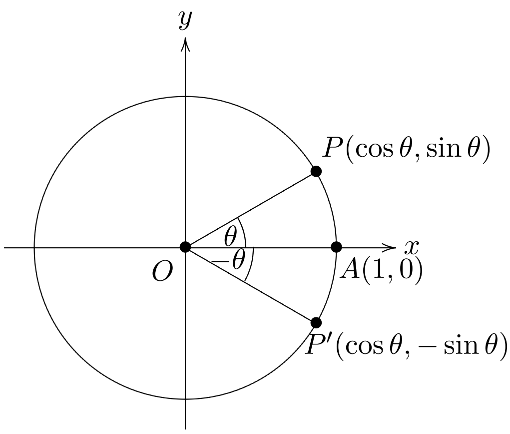
How to define $\cos\theta, \sin\theta$ **for other angles using simple geometry** If we know the values of $\sin\theta$ and $\cos\theta$ for angles $0 \le \theta \le \frac{\pi}{2}$ (or $0° \le \theta \le 90°$), then we also know them for arbitrary angles.



After one rotation, we are back at the point we started with, i.e., $P(\cos\theta, \sin\theta) = P(\cos(2\pi+\theta), \sin(2\pi+\theta))$. Therefore we have

$$\begin{aligned}\cos(2\pi+\theta) &= \cos\theta,\\ \sin(2\pi+\theta) &= \sin\theta.\end{aligned}$$

We also say that cosine and sine are "2π-periodic" or "have period 2π".

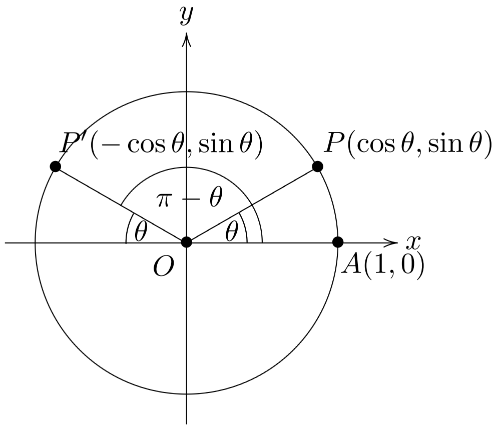


If we reflect P on the x-axis, we get a point P' with the same x-coordinate but negative y-coordinate. However, the point P' can also be otained by rotating through an angle $-\theta$ (or with the previous considerations, by $2\pi - \theta$). Thus, we have that P' also has coordinates $(\cos(-\theta), \sin(-\theta))$ and therefore

$$\begin{aligned}\cos(-\theta) &= \cos\theta,\\ \sin(-\theta) &= -\sin\theta.\end{aligned}$$

For tangent this yields:

$$\tan(-\theta) = -\tan\theta.$$

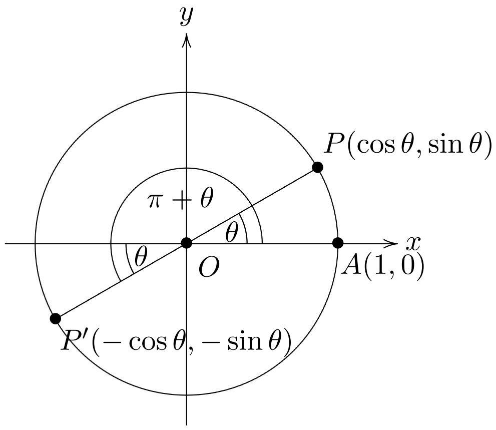


If we reflect P on the y-axis, we get a point P' with the same y-coordinate but negative x-coordinate. However, the point P' can also be otained by rotating through an angle $\pi - \theta$ (note that A is reflected to $(-1, 0)$). Thus, we have that P' also has coordinates $(\cos(\pi - \theta), \sin(\pi - \theta))$ and therefore

$$\begin{aligned} \cos(\pi - \theta) &= -\cos\theta, \\ \sin(\pi - \theta) &= \sin\theta. \end{aligned}$$

For tangent this yields:

$$\tan(\pi - \theta) = -\tan\theta.$$



If we reflect P in the origin, we get a point P' with both negative x- and y-coordinate. However, the point P' can also be otained by rotating through an angle $\pi + \theta$. Thus, we have that P' also has coordinates $(\cos(\pi + \theta), \sin(\pi + \theta))$ and therefore

$$\begin{aligned} \cos(\pi + \theta) &= -\cos\theta, \\ \sin(\pi + \theta) &= -\sin\theta. \end{aligned}$$

For tangent this yields:

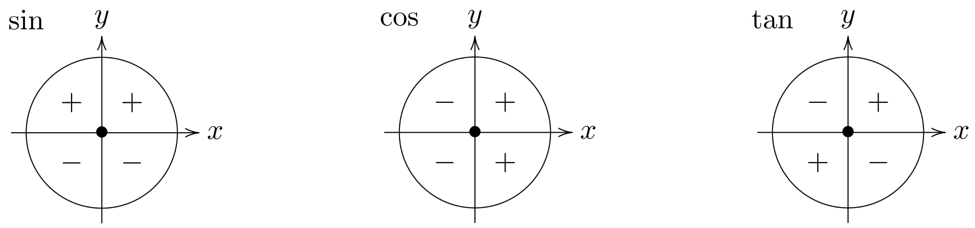
$$\tan(\pi + \theta) = \tan\theta.$$

The explicit calculation for the last observation is

$$\tan(\pi + \theta) = \frac{\sin(\pi + \theta)}{\cos(\pi + \theta)} = \frac{-\sin\theta}{-\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

This equation $\tan(\pi + \theta) = \tan\theta$ actually means that $\tan\theta$ is "π-periodic" respectively "has period π" (so we have, for example, that $\tan\frac{\pi}{7} = \tan\left(\pi + \frac{\pi}{7}\right) = \tan\left(2\pi + \frac{\pi}{7}\right)$ etc.).

Since $\sin\theta$ and $\cos\theta$ are positive for angles greater than 0 and less than $\frac{\pi}{2}$ (i.e., 90°), an easy consequence of our considerations are the following statements about the sign of the values $\sin\theta$, $\cos\theta$ and $\tan\theta$.



These pictures should be interpreted as

$$\sin\theta \text{ is } \begin{cases} \text{positive for angles } 0 < \theta < \frac{\pi}{2}, \\ \text{positive for angles } \frac{\pi}{2} < \theta < \pi, \\ \text{negative for angles } \pi < \theta < \frac{3\pi}{2}, \\ \text{negative for angles } \frac{3\pi}{2} < \theta < 2\pi, \end{cases}$$

and similarly for $\cos\theta$ and $\tan\theta$. Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

END OF LECTURE 10