MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 4

§5 Solving Equations

I. Quadratic Equations

After factorising quadratics using integers, we may ask: Which quadratics can be factorised (using reals) and how?
Quadratic equations are equations which involve x and x^2, where x is the unknown. There may be two, one or no solutions.

EXAMPLES:

(a) Solve $x^2 = 4$.
The solutions are $x = 2$ and $x = -2$ (equations of the type $x^2 = c$ with positive c have solutions $x = c$ and $x = -c$).

(b) Solve $x^2 - 3x + 1 = -1$.
Here, we are lucky and the equation can be solved by factorisation. It can be rewritten as

$$x^2 - 3x + 2 = 0$$

which is the same as

$$(x - 2)(x - 1) = 0$$

and this has solutions

$$x = 2 \text{ or } x = 1,$$

because for $x = 2$ the first factor on the left-hand side in the previous equation is zero, while for $x = 1$ the second factor is zero.

(c) Solve $x^2 - 4x + 1 = 2$.
This can be rewritten as $x^2 - 4x - 1 = 0$ for which, however, the left-hand side has no obvious integer factorisation (as we will see shortly, it really has no integer factorisation).
Instead, we write it as

$$x^2 - 4x = 1$$

Remembering that $x^2 - 4x + 4 = (x - 2)^2$ (recall: $(a \pm b)^2 = a^2 \pm 2ab + b^2$), we add the constant 4 to each side, so the left-hand side becomes a perfect square:

$$x^2 - 4x + 4 = 1 + 4$$

so

$$(x - 2)^2 = 5$$

and therefore

$$x - 2 = \sqrt{5} \text{ or } x - 2 = -\sqrt{5}.$$

Hence, the solutions of $x^2 - 4x + 1 = 2$ are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

Check: $(2 \pm \sqrt{5})^2 - 4(2 \pm \sqrt{5}) + 1 \; = \; 4 \cancel{\pm 4\sqrt{5}} + 5 \; - \; 8 \cancel{\mp 4\sqrt{5}} + 1 \; = \; 2$

This process is called "*completing the square*"!

MORE EXAMPLES:

	$2x^2 + 12x + 15 = 0$	$3x^2 + 4x - 5 = 0$
1) Divide by coefficient of x^2	$x^2 + 6x + \frac{15}{2} = 0$	$x^2 + \frac{4}{3}x - \frac{5}{3} = 0$
2) Put constant on right-hand side	$x^2 + 6x = -\frac{15}{2}$	$x^2 + \frac{4}{3}x = \frac{5}{3}$
3) Complete the square: Add $\left(\frac{1}{2} \times \text{coefficient of } x\right)^2$ to each side	$x^2 + 6x + (3)^2 = -\frac{15}{2} + 9$	$x^2 + \frac{4}{3}x \left(\frac{2}{3}\right)^2 = \frac{5}{3} + \frac{4}{9}$
4) left-hand side is a perfect square	$(x+3)^2 = \frac{3}{2}$	$\left(x + \frac{2}{3}\right)^2 = \frac{19}{9}$
5) Solve	$x = -3 \pm \sqrt{\frac{3}{2}}$	$x = -\frac{2}{3} \pm \frac{\sqrt{19}}{3}$

NOTE: This also yields the factorisation (to be precise, of the quadratic after step 1), i.e., after "*normalising*")

$$x^2 + 6x + \frac{15}{2} = \left(x - \left(-3 + \sqrt{\frac{3}{2}}\right)\right)\left(x - \left(-3 - \sqrt{\frac{3}{2}}\right)\right).$$

General quadratic equation

Instead of completing the square each time, we now look for a formula for solving a general quadratic

$$ax^2 + bx + c = 0.$$

Assume $a \neq 0$ (otherwise easy to solve $bx + c = 0$, namely $x = -\frac{c}{b}$ (well, as long as $b \neq 0$)).

We redo the above five steps for this general quadratic equation:

1) Divide by coefficient of x^2:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

2) Put constant on right-hand side:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

3) Add $\left(\frac{1}{2}\times \text{ coefficient of } x\right)^2$:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \qquad = \tfrac{b^2}{4a^2} - \tfrac{4ac}{4a^2}.$$

4) Left-hand side is a perfect square:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

5) Solve:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

hence

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The general formula for the solutions of a quadratic equation $ax^2 + bx + c = 0$ is:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Memorise this formula!

However, if you forget it, you can always obtain the solutions by completing the square!

END OF LECTURE 7

We have solved quadratic equations $ax^2 + bx + c = 0$ either by "completing the square" or by the general formula:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

We now have a closer look at this formula.

EXAMPLES

(i) Solve $2x^2 - 5x + 2 = 0$.

$$x = \frac{5 \pm \sqrt{5^2 - 4 \times 2 \times 2}}{2 \times 2} = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4} = 2 \text{ or } \frac{1}{2}$$

This equation has two solutions!

(ii) Solve $x^2 - 4x + 4 = 0$.

$$x = \frac{4 \pm \sqrt{16 - 16}}{2} = 2$$

This equation has one solution.
Also note that $x^2 - 4x + 4 = (x - 2)^2$ from which one can easily read off the solution $x = 2$.

(iii) Solve $x^2 - x + 3 = 0$.

$$x = \frac{1 \pm \sqrt{1 - 12}}{2} = \frac{1 \pm \sqrt{-11}}{2}$$

There is a negative number below the square root, so this equation has "no real solution", i.e., there is no real number that solves the equation $x^2 - x + 3 = 0$!

Summary

Consider the equation $ax^2 + bx + c = 0$.

Supposing $a \neq 0$, then the solution is $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

So, the equation has 2 solutions if $b^2 - 4ac > 0$
1 solution if $b^2 - 4ac = 0$
no solution if $b^2 - 4ac < 0$

The number $b^2 - 4ac$ is also called the *discriminant* of the quadratic $ax^2 + bx + c$.

ANOTHER EXAMPLE (NOT LECTURED):
Consider the equation $2x^2 + 12x + 15 = 0$ again (see Lecture 7).
The solutions are (as expected):

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 2 \times 15}}{2 \times 2} = \frac{-12 \pm \sqrt{144 - 120}}{4} = \frac{-12 \pm \sqrt{24}}{4} = -3 \pm \sqrt{\frac{3}{2}}.$$

Why quadratics? (not examinable)

Let us offer three reasons why we bother about quadratics and quadratic equations (we do not claim that this list is exhaustive):

(i) They appear frequently, especially (as a first approximation) if things are not linear.

(ii) One might also argue that one cares about quadratic equations because one can solve them without to much effort. For cubic and quartic equations (i.e., ones of the type $ax^3 + bx^2 + cx + d = 0$ and $ax^4 + bx^3 + cx^2 + dx + e = 0$), there exist also general formulas for the solutions, but they are awfully complicated. However, for a general quintic (i.e., $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$), no such formula for the solutions is actually possible!

(iii) Quadratic and linear factors appear naturally when factorising polynomials (i.e., expressions with powers of x, e.g., all quadratics, cubics, quartics etc. are polynomials): Every polynomial can be written as product of quadratic and linear factors with real coefficients (no cubic or quartic etc. factors are necessary!). For example, for the following randomly chosen quintics we have:

$$\begin{aligned}
x^5 - 2x^4 - x^3 + x^2 + 2x - 1 &= (x-1)(x-2.081\ldots)(x-0.480\ldots)\times \\
&\quad (x^2 + 1.561\ldots x + 1) \\
x^5 - x^4 + x^3 + x^2 + 2x - 1 &= (x-0.397\ldots)(x^2 - 1.974\ldots x + 2.436\ldots)\times \\
&\quad (x^2 + 1.371\ldots x + 1.033\ldots)
\end{aligned}$$

The quadratic factors here have no real solutions (also note that we give the real numbers here to 3 decimal places).
This implies that we can express any fraction of polynomials as partial fraction where the denominators are at most (powers of) quadratics! E.g., there are numbers A, B, C, D, E such that

$$\begin{aligned}
\frac{1}{x^5 - 2x^4 - x^3 + x^2 + 2x - 1} &= \frac{A}{x-1} + \frac{B}{x-2.081\ldots} + \frac{C}{x-0.480\ldots} + \\
&\quad \frac{Dx+E}{x^2 + 1.561\ldots x + 1}.
\end{aligned}$$

II. Simultaneous equations

EXAMPLE: Solve

$$\begin{aligned}
x + 2y &= 0 \qquad (1) \\
x - y &= 3 \qquad (2)
\end{aligned}$$

To solve we eliminate one of the unknowns: $2\times$ equation (2) gives

$$2x - 2y \;=\; 6. \tag{3}$$

Add (3) to (1) gives

$$(x + 2y) + (2x - 2y) \;=\; 0 + 6$$

We have eliminated the unknow y and have $3x = 6$ and hence $x = 2$. Substitute this in (1) to get

$$2 + 2y \;=\; 0,$$

so $y = -1$.
Therefore, the solution if the simulataneous equations (1) and (2) are $x = 2$ and $y = -1$.

Alternative solution: One can also directly subtract (2) from (1) which eliminates x and gives

$$(x + 2y) - (x - y) \;=\; 0 - 3,$$

thus $3y = -3$. Of course, we again have $y = -1$ and substituting this in (1) (or (2)) one obtains $x = 2$ as before.

Another Example: Solve

$$\begin{aligned} 2x + 3y &= 13 && (1)\\ x - y &= -1. && (2) \end{aligned}$$

$3\times$ (2) gives

$$3x - 3y \;=\; -3. \tag{3}$$

Add (1) to (3) gives

$$5x = 10.$$

Hence $x = 2$ and – substituting this in (1) to get $3y = 9$ – also $y = 3$.
So, the solution of the simulataneous equations $2x + 3y = 13$ and $x - y = -1$ is $x = 2$ and $y = 3$.

Three simultaneous equations und three unknowns (not lectured, not examinable)

This can be solved by eliminating first one of the unknowns and then in a second step the second unknow. For example, if one has

$$\begin{aligned} x + y + z &= 1 && (1)\\ -2x + y - 3z &= 0 && (2)\\ 3x + y &= 2. && (3) \end{aligned}$$

In a first step, we eliminate x. For this, we add $2\times$ (1) to (2) and we subtract (3) from $3\times$ (1) to obtain

$$\begin{aligned} 2(x+y+z)+(-2x+y-3z) &= 2+0, \quad \text{i.e.,} \quad 3y-z = 2 \qquad (4)\\ 3(x+y+z)-(3x+y) &= 3-2, \quad \text{i.e.,} \quad 2y+3z = 1 \qquad (5)\end{aligned}$$

With (4) and (5) we proceed as before, e.g., subtracting $3\times$ (5) from $2\times$ (4) eliminates y and yields

$$2(3y-z)-3(2y+3z) = 4-3, \quad \text{i.e.,} \quad -11z = 1.$$

So, we have $z = -\frac{1}{11}$, and substituting this in (4) yields $3y + \frac{1}{11} = 2$ and therefore $y = \frac{7}{11}$. Now, substituting $z = -\frac{1}{11}$ and $y = \frac{7}{11}$ in (1) yields

$$x + \frac{7}{11} - \frac{1}{11} = 1$$

and hence $x = \frac{5}{11}$.
Consequently, the solution of the simultaneous equations (1)–(3) is $x = \frac{5}{11}$, $y = \frac{7}{11}$ and $z = -\frac{1}{11}$.

We now "combine" quadratic and simultaneous equations as follows:

Example: Solve

$$\begin{aligned} x-y &= 2 \qquad (1)\\ x^2+y^2 &= 10 \qquad (2)\end{aligned}$$

The first equation is linear, while the second is quadratic. Here we can use (1) to express x in terms of y and then substitute into (2) to get a quadratic equation in y only.
Rewrite (1) to get $x = y+2$.
Substituting this into (2) we have

$$10 = (y+2)^2 + y^2 = 2y^2 + 4y + 4$$

and therefore $2y^2 + 4y - 6 = 0$. The solutions of this quadratic equation are

$$y = \frac{-4 \pm \sqrt{16+48}}{4} = \frac{-4 \pm \sqrt{64}}{4} = -1 \pm 2,$$

i.e., $y = -3$ or $y = 1$.
Now substitute $y = 1$ in (1) which gives $x = 3$, while $y = -3$ in (1) gives $x = -1$
So the solutions are: $x = -1, y = -3$ and $x = 3, y = 1$.

Another example (not lectured)
Solve

$$
\begin{aligned}
x - y &= 2 \quad (1)\\
2x^2 - 3y^2 &= 15. \quad (2)
\end{aligned}
$$

From (1), $x = y + 2$. Substitute this into (2) to get:

$$2(y+2)^2 - 3y^2 = 15.$$

This can be rewritten as $-y^2 + 8y - 7 = 0$ or $y^2 - 8y + 7 = 0$. Factorising this gives $(y-7)(y-1) = 0$, and so $y = 7$ and $y = 1$. Substituting into (1) we get solutions: $x = 9,\ y = 7$ and $x = 3,\ y = 1$

END OF LECTURE 8