MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 3

Indices/Powers

In an expression a^n, a is called the *base* and n is called the *index* (or *power* or *exponent*).

Multiplication/Division of Powers

$$a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7$$

This illustrates:

$$\boxed{\text{Rule 1: } a^p \times a^q = a^{p+q}}$$

Also: $a^5 \div a^4 = \dfrac{a \times a \times a \times a \times a}{a \times a \times a \times a} = a$

$$\boxed{\text{Rule 2: } a^p \div a^q = a^{p-q}}$$

Consequences of Rule 2:

$$\frac{a}{a} = \frac{a^1}{a^1} = a^{1-1} = a^0. \text{ But also } \frac{a}{a} = 1$$

so $\boxed{a^0 = 1}$

Also $\frac{1}{a^p} = \frac{a^0}{a^p} = a^{0-p} = a^{-p}$

so $\boxed{\dfrac{1}{a^p} = a^{-p}}$

Powers of powers

EXAMPLES:

$$\begin{aligned}(a^3)^2 &= a^3 \times a^3 = a^6 \\ (a^2)^5 &= a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{10}\end{aligned}$$

$$\boxed{\text{Rule 3: } (a^p)^q = a^{pq}}$$

NOTE: The brackets are important here! a^{p^q} is $a^{(p^q)}$ but not $(a^p)^q$ (therefore, to reduce the chance of confusion, never use a^{p^q}. Always use the variant with brackets!). E.g., $10^{3^2} = 10^{(3^2)} = 10^9$ (a billion), but $(10^3)^2 = 10^{3\times 2} = 10^6$ (a million).

The numbers p, q in Rule 3 do not have to be integers, so

$$\left(a^{\frac{1}{n}}\right)^n = a^1 = a.$$

This shows:

$$\boxed{\text{Rule 4: } \sqrt[n]{a} = a^{\frac{1}{n}}}$$

NOTE: If in an expression a^p the number p is not an integer, then a has to be positive.

EXAMPLES

(i) Evaluate $4^{\frac{2}{3}}$.

$$\begin{aligned} 4^{\frac{2}{3}} &\overset{\text{Rule 3}}{=} \left(4^2\right)^{\frac{1}{3}} = 16^{\frac{1}{3}} \\ &\overset{\text{Rule 4}}{=} \sqrt[3]{16} = \sqrt[3]{8 \times 2} = 2\sqrt[3]{2}. \end{aligned}$$

(ii) Evaluate $4^{\frac{3}{2}}$.

$$\begin{aligned} 4^{\frac{3}{2}} &\overset{\text{Rule 3}}{=} \left(4^{\frac{1}{2}}\right)^3 \\ &\overset{\text{Rule 4}}{=} \left(\sqrt{4}\right)^3 = 2^3 = 8. \end{aligned}$$

(iii) Evaluate $(4^3)^{\frac{1}{2}}$.

$$\left(4^3\right)^{\frac{1}{2}} \overset{\text{Rule 3}}{=} 4^{\frac{3}{2}} \overset{\text{see (ii)}}{=} 8.$$

(iv) Simplify $\dfrac{2 \times 2^3}{4^2}$.

$$\frac{2 \times 2^3}{4^2} \overset{\text{Rule 1}}{=} \frac{2^4}{4^2} \overset{4=2^2}{=} \frac{2^4}{\left(2^2\right)^2} \overset{\text{Rule 3}}{=} \frac{2^4}{2^4} \overset{\text{Rule 2}}{=} 2^{4-4} = 2^0 = 1$$

Logarithms

EXAMPLES: (a) What is the power that 10 must be raised to, to give answer 100?
Answer: Since $10^2 = 100$, answer is 2.

(b) What power must 2 be raised to, to give 16?
Answer: 4

We can express these results using logarithms:

(a) says $\log_{10} 100 = 2$

(b) says $\log_2 16 = 4$

In general:

$$\boxed{b = a^c \text{ is equivalent to } \log_a b = c.}$$

In words: *"The logarithm of b to the base a is c."*

EXAMPLE: What is $\log_4 64$?
Since $4^3 = 64$, one has $\log_4 64 = 3$.

Manipulating logarithms

How can we simplify expressions like $\log_a(bc)$? The rule is:

$$\boxed{\text{Rule 1: } \log_a(bc) = \log_a b + \log_a c}$$

Where does this rule come from?

Suppose $x = \log_a b$ and $y = \log_a c$. This means $a^x = b$ and $a^y = c$.
So, from Rule 1 for indices (recall $a^p \times a^q = a^{p+q}$) we have $b \times c = a^x \times a^y = a^{x+y}$. We can write this another way as $\log_a(bc) = x + y$, and therefore obtain $\log_a(bc) = x + y = \log_a b + \log_a c$.
So we have proved Rule 1.

Similarly, one can show (using Rule 2 for indices, i.e., $a^p/a^q = a^{p-q}$):

$$\boxed{\text{Rule 2: } \log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c}$$

and (using Rule 3 for indices, namely $(a^p)^q = a^{pq}$)

$$\boxed{\text{Rule 3: } \log_a(b^d) = d \log_a b}$$

NOTE: Since log is the inverse operation to taking powers, the rules for manipulating logarithms can be deduced from the rules for manipulating indices here.

EXAMPLE: Express $\log_a\left(\frac{p}{\sqrt{q}}\right)$ in terms of $\log_a p$ and $\log_a q$.

$$\log_a\left(\frac{p}{\sqrt{q}}\right) \overset{\text{Rule 1}}{=} \log_a p - \log_a\left(q^{\frac{1}{2}}\right) \overset{\text{Rule 3}}{=} \log_a p - \frac{1}{2}\log_a q.$$

END OF LECTURE 5

In this chapter we have looked at expressions of the type $a^p = b$. Note that the following cases may emerge:

- You know p and b. Then a is given by $a = \sqrt[p]{b}$.
- You know a and p. Then b is given by $b = a^p$.
- You know a and b. Then p is given by $p = \log_a b$.

We summarise the rules for manipulating surds, indices and logarithms here:

- Surds:
  - $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$
  - $\sqrt[n]{\frac{a}{b}} = \sqrt[n]{a} \div \sqrt[n]{b}$
- Indices:
  - Rule 1: $a^p \times a^q = a^{p+q}$
  - Rule 2: $a^p \div a^q = a^{p-q}$
  - Rule 3: $(a^p)^q = a^{pq}$
  - Rule 4: $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^0 = 1$ and $a^{-p} = \frac{1}{a^p}$.
- Logarithms:
  - Rule 1: $\log_a(bc) = \log_a b + \log_a c$
  - Rule 2: $\log_a \frac{b}{c} = \log_a b - \log_a c$
  - Rule 3: $\log_a b^d = d \times \log_a b$

Further observations (for logarithms):

(a) Since $x^0 = 1$, this can be rewritten as $\boxed{\log_x 1 = 0}$ (for any $x \neq 0$).

Think about the condition $x \neq 0$: What would "$\log_0 a$" mean, and for which numbers a is this meaningful?

(b) Also $\log_a \left(\frac{1}{c}\right) \overset{\text{Rule 2}}{=} \log_a 1 - \log_a c$, so $\boxed{\log_a \left(\frac{1}{c}\right) = -\log_a c}$

Example

Write $-\log_a p + 3\log_a q$ as a single logarithm.

$$-\log_a p + 3\log_a q \overset{\text{Rule 3}}{=} \log_a \left(\frac{1}{p}\right) + \log_a \left(q^3\right) \overset{\text{Rule 1}}{=} \log_a \left(\frac{q^3}{p}\right)$$

Natural logarithm
There is a special irrational number that plays an important role as base in calculations involving logarithms and powers (especially, for integration and differentiation of functions, something we will about later in this course): Euler's number $\mathrm{e} = 2.71828\ldots$.

Powers to this base e are written as $\mathrm{e}^{\mathrm{x}} = \mathrm{y}$, while logarithms – which should read $\log_{\mathrm{e}} y = x$ – are written $\ln y = x$ (ln is the abbreviation of the Latin "**l**ogarithmus **n**aturalis").

The natural logarithm ln and the logarithm to base 10, which is abbreviated log (written without any base!), can be found on a (scientific) calculator. This notation for logarithms is also used in applied sciences.

WARNING: In (pure) mathematics, however, log usually denotes the natural logarithm ($\log_{10}$ plays no special role there!).
In this course, $\log = \log_{10}$ **always!**

Calculating logarithms
Look at the following table:

x	1	2	5	10
$\log x$	0	0.301 (3 d.p.)	0.699 (3 d.p.)	1

If one remembers $\log 2 \approx 0.3$, many logarithms can be estimated closely.

Examples: $\log 4 = \log(2 \times 2) = \log 2 + \log 2 \approx 0.3 + 0.3 = 0.6$
(in fact, $\log 4 = 0.602$ (3 d.p.))
$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 \approx 1 - 0.3 = 0.7$ (in fact, $\log 5 = 0.699$ (3 d.p.))
$\log 8 = \log 2^3 = 3 \log 2 \approx 3 \times 0.3 = 0.9$ (in fact, $\log 8 = 0.903$ (3 d.p.))

More examples on logarithms

(a) If $10^x = 5$, find x.
$x = \log_{10}(5) = 0.699$ (3 d.p.) (see above or use your calculator).

(b) If $(0.1)^x = 5$, find x.
The easy solution would be $x = \log_{0.1} 5$, but then, how do we calculate logarithms to base 0.1? Note that $0.1 = \frac{1}{10} = 10^{-1}$, therefore

$$5 = (0.1)^x = \left(10^{-1}\right)^x = 10^{-x},$$

so $-x = \log 5$ and hence $x = -\log 5 = -0.699$ (3 d.p.).

Change of base
In more general terms, the last example (b) asks the following question: Given a^p and a number b, can one find the index q such that $a^p = b^q$?

Write $a = b^x$, then $x = \log_b a$, i.e., $a = b^{\log_b a}$ (this is actually the definition of the logarithm!). But then

$$a^p = \left(b^{\log_b a}\right)^p = b^{p \log_b a},$$

i.e.,

$$\boxed{a^p = b^{p \log_b a}} \quad (\circledast).$$

On the calculator, only ln and log can be found. How can we calculate $\log_2 5$, $\log_3 7$ etc.?

Again, $p = \log_a c$ means $c = a^p \overset{\circledast}{=} b^{p \log_b a}$, hence $\log_b c = p \log_b a$. So, we have two equations for p, namely $p = \log_a c$ and $p = \log_b c / \log_b a$, and therefore obtain

$$\boxed{\log_a c = \frac{\log_b c}{\log_b a}}$$

This is the formula if we want to change the base in logarithms (the "old" base a appears in the logarithm in the denominator on the right-hand side!)

EXAMPLE: $\log_2 5 = \frac{\log 5}{\log 2} = 2.322$ (3 d.p.)
With $\log 2 \approx 0.3$ and $\log 5 \approx 0.7$, we estimate $\log 5 \div \log 2 \approx 0.7/0.3 = \frac{7}{3} = 3.\overline{3}$.

More examples and a warning (not lectured)

EXAMPLES:

(a) If $3^x = 5$, find x.

$$x = \log_3 5 = \frac{\log 5}{\log 3} = 1.465 \text{ (3 d.p.)}.$$

(b) Write $\log_8 32$ in terms of logs to base 2, hence find $\log_8 32$ exactly.

$$\log_8 32 = \frac{\log_2 32}{\log_2 8} = \frac{5}{3}$$

WARNING: $\frac{\log_b c}{\log_b a} \neq \log_b \frac{c}{a}$ (e.g., $\log 10 / \log 2 = 1/\log 2 = 3.322$ (3 d.p.), while $\log \frac{10}{2} = \log 5 = 0.699$ (3 d.p.))
Expression where two logarithms are multiplied or divided cannot be simplified, at least not in an "easy way"!
For the "not-so-easy" way, study the following example:

$$(\log_2 100) \times (\log 2) \overset{\text{Rule 3 for log}}{=} \log\left(2^{\log_2 100}\right) = \log 100 = 2.$$

END OF LECTURE 6