MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 2

EXAMPLE REVISITED: Factorise $x^2+7x+12$.
We are looking for a, b such that $x^2+7x+12=(x+a)(x+b)=x^2+(a+b)x+ab$. We have $a+b=7$ and $ab=12$, moreover, from these we also know that both a and b are positive. What are the possibilities to have $ab=12$?

a	b	$a+b$
1	12	13
2	6	8
3	4	7
4	3	7
6	2	8
12	1	12

So, from $a+b=7$, we obtain the solution $a=3$ and $b=4$ (or vice versa, $a=4$ and $b=3$) and therefore

$$x^2+7x+12=(x+3)\,(x+4).$$

EXAMPLE. Factorise $2x^2-7x+3$.
This is harder: Since the coefficient of x^2 is not equal to 1, we cannot use the previous method. Certainly, the only possibility with integer coefficients is $(2x+a)(x+b)=2x^2+(a+2b)x+ab$. Now, go through pairs with product $ab=3$ (note that both a and b must be negative (why?)) until we find the one that works:

$$(2x-3)(x-1)=2x^2-3x-2x+3=2x^2-5x+3 \text{ doesn't work.}$$

$$(2x-1)(x-3)=2x^2-6x-x+3=2x^2-7x+3 \text{ works.}$$

So $2x^2-7x+3=(2x-1)(x-3)$.

EXAMPLE. Factorise $6x^2+13x+6$. Possibilities are $(6x+a)(x+b)$ or $(3x+a)(2x+b)$. As before, we can deduce that a and b are both positive. The constant term is ab for both possibilities.
Try $(6x+1)(x+6), (6x+6)(x+1), (6x+2)(x+3), (6x+3)(x+2)$ find that they don't work (do this!). Eventually $(3x+2)(2x+3)$ works.

NOTE: Not all quadratics can be factorised using integer coefficients, e.g., x^2+1 or x^2-x-1. However, some of these can be factorised using real numbers, e.g.,

$$x^2-x-1=\left(x-\frac{1+\sqrt{5}}{2}\right)\left(x-\frac{1-\sqrt{5}}{2}\right).$$

We will look at such examples later in the course.

§3 Fractions

Fractions with letters work exactly like fractions with numbers. So, to simplify a fraction, look for common factors of numerator and denominator.
EXAMPLE:

$$\frac{8a^2+8ab}{10ab+10b^2} \overset{\text{factorise}}{=} \frac{8a(a+b)}{10b(a+b)} \overset{\text{cancel}}{=} \frac{4a}{5b}$$

To add fractions we seek a common denomiator:
EXAMPLE:

$$\frac{1}{(x+1)} + \frac{1}{(x-2)} = \frac{(x-2)}{(x+1)(x-2)} + \frac{(x+1)}{(x+1)(x-2)}$$

$$\text{So} \quad \frac{1}{(x+1)} + \frac{1}{(x-2)} = \frac{2x-1}{(x+1)(x-2)} \qquad (*)$$

Above we have expressed the sum of two fractions as a single fraction. The question now is: Given the right-hand side of (*), could we reconstruct the simpler fractions on the left-hand side which sum to this? This is called "*expressing in partial fractions*".

EXAMPLE: Express in partial fractions: $\dfrac{2x+7}{(x-1)(x+2)}$.

Write as $\dfrac{2x+7}{(x-1)(x+2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+2)}$.

To find A, B, sum the right-hand side and force it to equal the left-hand side:

$$\frac{2x+7}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} = \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x+2)(x-1)}.$$

Fractions with the same denominator are equal, if their numerators are equal. So, we equat the numerators here:

$$2x+7 = A(x+2) + B(x-1).$$

This has to hold for all x. To find A, B we can choose particular convenient values of x.

$x = -2:$ gives $-3B = 3$ and hence $B = -1$
$x = 1:$ gives $3A = 9$ and hence $A = 3$

So $\dfrac{2x+7}{(x-1)(x+2)} = \dfrac{3}{(x-1)} - \dfrac{1}{(x+2)}$

EXAMPLE: Express in partial fractions: $\dfrac{6x+1}{(2x-3)(x+1)}$.

As before, we start with

$$\begin{aligned}\frac{6x+1}{(2x-3)(x+1)} &= \frac{A}{2x-3} + \frac{B}{x+1} \\ &= \frac{A(x+1)+B(2x-3)}{(2x-3)(x+1)}\end{aligned}$$

Equating the numerators gives

$$6x+1 = A(x+1) + B(2x-3).$$

This has to hold for all x, so we can choose some particular values of x to get easy equations for A and B:

$$\begin{array}{llrcl} x=-1: & \text{gives} & -5 = -5B & \text{and hence} & B = 1 \\ x=\frac{3}{2}: & \text{gives} & 10 = 6\times\frac{3}{2}+1 = \frac{5}{2}A & \text{and hence} & A = 4 \end{array}$$

So $\dfrac{6x+1}{(2x-3)(x+1)} = \dfrac{4}{(2x-3)} + \dfrac{1}{(x+1)}$.

END OF LECTURE 3

EXAMPLE: Express in partial fractions $\frac{x+2}{(3x+2)(x+1)}$.
The first step is the following:

$$\frac{x+2}{(3x+2)(x+1)} = \frac{A}{3x+2} + \frac{B}{x+1} = \frac{A(x+1)+B(3x+2)}{(3x+2)(x+1)}.$$

The denominators on the left- and right-hand side are the same. Look at the numerators (we want to have that the fractions on the left- and right-hand side are the same), this yields

$$x+2 = A(x+1) + B(3x+2). \qquad (\star)$$

Recall that we are looking for the values of A and B (while x is a variable, and therefore "x can vary", i.e., the last equation has to hold for every value you use for x). There are two methods.

Method I: Expand the right-hand side of $(\star)$.
The left-hand side equals the right-hand side in $(\star)$ if we have the same polynomial on both sides, i.e., the coefficients have to be equal:

$$x+2 = A(x+1)+B(3x+2) = Ax+A+3Bx+2B = (A+3B)x+(A+2B)$$

so the coefficient of the x-term is 1 on the left and $A+3B$ on the right, while the constant is 2 on the left while it is $A+2B$ on the right. Therefore, we have to find A and B such that the following two equations are satisfied:

$$1 = A+3B \qquad\qquad 2 = A+2B$$

We will at such simultaneous equations later in this course.

Method II: The left-hand side equals the right-hand side in $(\star)$ if they are equal for every value of x. So, we "test" this equality for some particular (well-chosen) values of x.
At $x=-1$ we are left with an expression for B since the A-term vanishes:

$$-1+2 = 1 \;=\; A(-1+1)+B(3\times(-1)+2) = \not{A}\not{\times}\not{0} + B\times(-1)$$

and hence $B=-1$.
At $x=-\frac{2}{3}$ the same phenomena appears with the roles of A and B interchanged:

$$-\frac{2}{3}+2 = \frac{4}{3} \;=\; A(-\frac{2}{3}+1)+B(3\times(-\frac{2}{3})+2) = A\frac{1}{3} + \not{B}\not{\times}\not{0}$$

and hence $A=4$.

Note: Instead of evaluating the above equation at $x=-\frac{2}{3}$, you can – since you already know $B=-1$ – use any value for x, e.g., $x=0$ in $(\star)$ yields $2 = A\times 1 + B\times 2 \overset{B=-1}{=} A-2$ and hence again $A=4$. Or if you use $x=10$ in $(\star)$ you obtain $12 = A\times 11 + B\times 32 \overset{B=-1}{=} A\times 11 - 32$ and therefore again $A=4$.

Note: $A=4$ and $B=-1$ is, of course, also the solution of the simultaneous equations in Method I above.

We have obtained
$$\frac{x+2}{(3x+2)(x+1)} = \frac{4}{3x+2} + \frac{-1}{x+1}.$$

Examples & Remarks (not given in the lecture, not examinable)

EXAMPLE: Express in partial fractions $\frac{1}{(6x^2+13x+6)(x+1)}$.
As a first step, check if the denominator is factorisable.
$$\frac{1}{(6x^2+13x+6)(x+1)} \overset{\text{see Lecture 3}}{=} \frac{1}{(3x+2)(2x+3)(x+1)}.$$
There are now three linear factors in the denominator. Therefore, we look for the values of A, B, and C in the following equation, and proceed as before:
$$\begin{aligned}\frac{1}{(3x+2)(2x+3)(x+1)} &= \frac{A}{3x+2} + \frac{B}{2x+3} + \frac{C}{x+1} \\ &= \frac{A(2x+3)(x+1) + B(3x+2)(x+1) + C(3x+2)(2x+3)}{(3x+2)(2x+3)(x+1)}.\end{aligned}$$
We again look at the numerators, which yields the equation $1 = A(2x+3)(x+1) + B(3x+2)(x+1) + C(3x+2)(2x+3)$. Evaluating this last equation at $x = -\frac{2}{3}$ yields $A = \frac{9}{5}$, evaluating it at $x = -\frac{3}{2}$ yields $B = \frac{4}{5}$ and evaluating at $x = -1$ yields $C = -1$ (check this).

REMARK: If a quadratics is not factorisable, the numerator can be linear in x, e.g., for the following fraction we look for A, B and C (i.e., A and $Bx + C$) on the right-hand side
$$\frac{x+1}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3} = \frac{A(x^2+3) + (Bx+C)(x-1)}{(x+1)(x^2+3)}.$$
This can be solved using "Method I" above (which leads to simultaneous equations). The solution is $A = \frac{1}{2}$, $B = -\frac{1}{2}$ and $C = \frac{1}{2}$.

REMARK: If a factor in the denominator is a square (or some power) and one wants to express this as partial fraction, also the linear term appears as a denominator (respectively all powers up to the power in question), e.g., for the following fraction we look for A, B and C:
$$\frac{x^2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}.$$
Equating the numerators and evaluating them at $x = 1$ and $x = -1$ yields $C = \frac{1}{2}$ and $A = \frac{1}{4}$. Using this knowledge and evaluating at any other value for x yields $B = \frac{3}{4}$. So,
$$\frac{x^2}{(x+1)(x-1)^2} = \frac{1}{4(x+1)} + \frac{3}{4(x-1)} + \frac{1}{2\,(x-1)^2}.$$

Remark: If the highest power of x in the numerator is higher or equal to the highest power of x in the denominator, one can extract a "whole" term (meaning something that is not a fraction), e.g., in the following the highest power of x in the numerator and the denominator is 2:

$$\frac{x^2}{(x-1)(x+1)} = \frac{x^2}{x^2-1} = \frac{x^2-1+1}{x^2-1} = 1 + \frac{1}{x^2-1} = 1 + \frac{1}{(x-1)(x+1)}$$
$$\overset{\text{partial fract.}}{=} 1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

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§4 Surds, Indices & Logarithms

Square roots
If $b = a \times a$, then we say that a is the square root of b, written $a = \sqrt{b}$, e.g., $2 = \sqrt{4}$.
$\sqrt{b}$ always means the postive square root of b.
Example $\sqrt{49} = 7$ and $-7 = -\sqrt{49}$

General roots
Examples:

$$16 = 2 \times 2 \times 2 \times 2 \quad \text{so} \quad 2 = \sqrt[4]{16} \quad (\text{"the fourth root"})$$

$$1024 = 4 \times 4 \times 4 \times 4 \times 4 \quad \text{so} \quad 4 = \sqrt[5]{1024} \quad \text{Also } 2 = \sqrt[10]{1024}\ .$$

If $b = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$, then $a = \sqrt[n]{b}$.

Surds
Some roots are irrational, e.g., $\sqrt{2}$. Such numbers are called *surds* and are manipulated as symbols.

Irrationality of $\sqrt{2}$ (not given in lecture, not examinable)

This is an example in mathematical reasoning.
One easily verifies the following two statements:

(a) If a fraction $\frac{n}{m}$ is in lowest terms, the numerator n and the denominator m cannot be both even numbers (if they were, we could cancel a common factor of 2).

(b) The square of an even number is even, the square of an odd number is odd.

Now assume that $\sqrt{2}$ is rational, i.e., there are integers n and m such that $\sqrt{2} = \frac{n}{m}$. Let $\frac{n}{m}$ be in lowest terms (so, according to (a) above, at least one of the two numbers is odd). Taking the square of $\sqrt{2} = \frac{n}{m}$ on both sides yields $2 = \frac{n^2}{m^2}$ and hence $2m^2 = n^2$. But this shows, that n^2 contains a factor of 2 and is therefore even. Using (b) above, we therefore conclude that n itself is even; consequently, m is odd.
So far, we have: If we assume that $\sqrt{2} = \frac{n}{m}$ is rational, then the numerator n is even and the denominator m is odd.

However, since n is even, there is another integer k such that $n = 2k$ (k is half of n and an integer since n is even). Then $\sqrt{2} = \frac{n}{m} = \frac{2k}{m}$ and squaring this yields $2 = \frac{4k^2}{m^2}$ and hence $m^2 = 2k^2$. But then, m^2 (being twice some integer) is even and therefore (using (b)) also m!

Result: From the assumption that $\sqrt{2}$ is rational, $\sqrt{2} = \frac{n}{m}$, we have concluded that the denominator m is both even and odd. This is absurd, since an integer is either even or odd, therefore the assumption has to be wrong. So, $\sqrt{2}$ is not rational.

Addition/Subtraction
EXAMPLE: $\sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$.

We cannot simplify $\sqrt{2} - 5\sqrt{3}$.

Multiplication
The rule is: $\boxed{\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}}$
EXAMPLES:

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$$

$$\sqrt[3]{24} = \sqrt[3]{8 \times 3} = \sqrt[3]{8} \times \sqrt[3]{3} = 2\sqrt[3]{3}$$

Division
The rule is : $\boxed{\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}}$
EXAMPLE: $\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$

Examples

(a) Simplify $\sqrt{12} - \sqrt{3} + \sqrt{5}$.
Reducing to smallest possible surds, answer is

$$\sqrt{12} - \sqrt{3} + \sqrt{5} = 2\sqrt{3} - \sqrt{3} + \sqrt{5} = \sqrt{3} + \sqrt{5}.$$

(b) Simplify: $(2+\sqrt{3})(3+\sqrt{5})$.

$$\begin{aligned}(2+\sqrt{3})(3+\sqrt{5}) &= 2\times 3+2\sqrt{5}+3\sqrt{3}+\sqrt{3}\times\sqrt{5}\\ &= 6+2\sqrt{5}+3\sqrt{3}+\sqrt{15}.\end{aligned}$$

(c) Simplify: $\sqrt{7}(4-\sqrt{7})$.

$$\sqrt{7}(4-\sqrt{7}) = 4\sqrt{7}-\sqrt{7}\sqrt{7} = 4\sqrt{7}-7.$$

When simplifying fractions we are required to make sure that the denominator is rational.

EXAMPLE:

$$\underset{\substack{\uparrow \\ \text{irrational denom.}}}{\frac{3}{\sqrt{5}}} = \frac{3\sqrt{5}}{\sqrt{5}\sqrt{5}} = \underset{\substack{\uparrow \\ \text{rational denom.}}}{\frac{3\sqrt{5}}{5}}$$

For fractions such as $\frac{2}{4-\sqrt{3}}$, multiplying numerator and denominator by $4+\sqrt{3}$ will rationalise the denominator (remember the formula for the difference of two squares: $(a-b)(a+b)=a^2-b^2$):

$$\frac{2}{4-\sqrt{3}} = \frac{2(4+\sqrt{3})}{(4-\sqrt{3})(4+\sqrt{3})} = \frac{8+2\sqrt{3}}{16-3} = \frac{8+2\sqrt{3}}{13}\ .$$

EXAMPLES:

(a) Simplify $\frac{\sqrt{5}}{3\sqrt{7}-2}$.

$$\frac{\sqrt{5}}{3\sqrt{7}-2} = \frac{\sqrt{5}(3\sqrt{7}+2)}{(3\sqrt{7}-2)(3\sqrt{7}+2)} = \frac{3\sqrt{5}\sqrt{7}+2\sqrt{5}}{9(\sqrt{7})^2-4} = \frac{3\sqrt{35}+2\sqrt{5}}{63-4} = \frac{3\sqrt{35}+2\sqrt{5}}{59}$$

(b) Simplify $\frac{\sqrt{5}}{\sqrt{5}-1}$.

$$\frac{\sqrt{5}}{\sqrt{5}-1} = \frac{\sqrt{5}(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{(\sqrt{5})^2+\sqrt{5}}{(\sqrt{5})^2-1} = \frac{5+\sqrt{5}}{5-1} = \frac{5+\sqrt{5}}{4}$$

END OF LECTURE 4