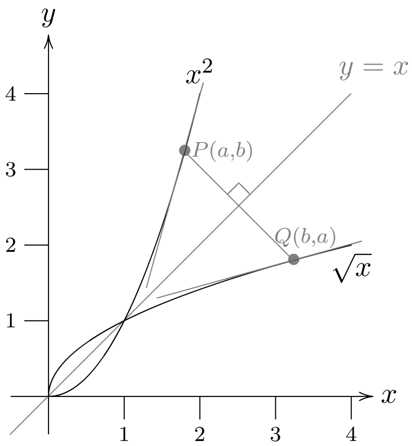
MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 11

Inverse functions and derivatives

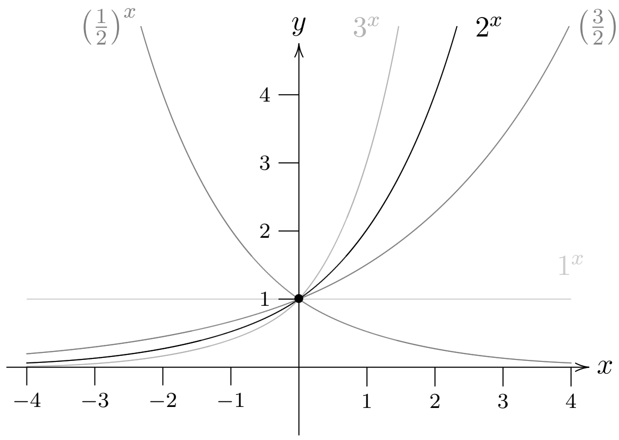
If we consider the curve $y = f(x)$, then the curve $y = g(x)$ of its inverse is obtained by reflecting in the line $y = x$. Reflecting a point $P(a, b)$ on $y = x$ yields the point $Q(b, a)$. E.g., if P lies on $y = x^2$, then $b = a^2$, i.e., $P(a, a^2)$. Rewriting this, we have $a = \sqrt{b}$ and the reflected point Q has coordinates $(a^2, a) = (b, a) = (b, \sqrt{b})$, wherefore the reflected curve is indeed $y = \sqrt{x}$.



Note that we also have a relationship between the derivatives: If the tangent in $P(a, b)$ has gradient $f'(a)$, then the corresponding gradient at $Q(b, a)$ is $g'(b) = \frac{1}{f'(a)}$. E.g., for $f(x) = x^2$ we know that the derivative at $x = a$ is $f'(a) = 2a = 2\sqrt{b}$. Consequently, the derivative of $g(x) = \sqrt{x}$ at b is $g'(b) = 1/f'(a) = 1/(2\sqrt{b})$ (which shows that $g'(x) = \frac{1}{2\sqrt{x}}$).

The exponential function

Consider the function $y = a^x$ for various $a > 0$. The graphs look like:



The choice of a that makes the gradient at $x = 0$ equal 1 is called e. (e = 2.71828 (5 d.p.)). That is, we have

$$f'(0) = 1 \quad \text{for} \quad f(x) = \mathrm{e}^x, \qquad \text{i.e.,} \qquad \frac{\mathrm{e}^{\delta x} - \mathrm{e}^0}{\delta x} \to 1 \qquad \text{as} \quad \delta x \to 0.$$

Now consider any other x and calculate the derivative:

$$\frac{e^{x+\delta x} - e^x}{\delta x} = \frac{e^x \times e^{\delta x} - e^x}{\delta x} = e^x \left(\frac{e^{\delta x} - 1}{\delta x}\right) \to e^x \times 1 \qquad \text{as} \quad \delta x \to 0.$$

Hence for any x:

$$\boxed{\frac{d}{dx} e^x = e^x}$$

EXAMPLES:

(a) $\frac{d}{dx}\left(e^{x+1}\right) = \frac{d}{dx}(e\, e^x) = e \frac{d}{dx} e^x = e^{x+1}$.

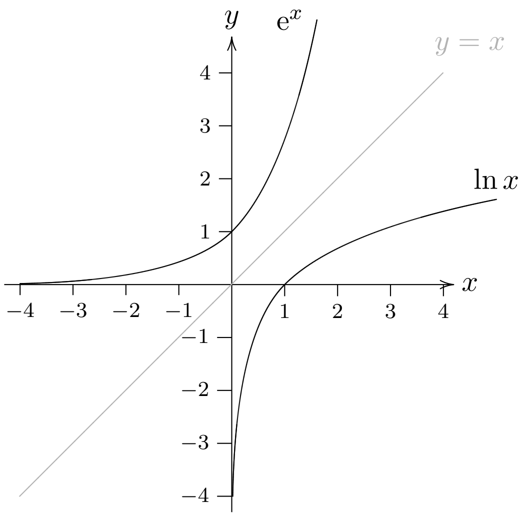
(b) $\frac{d}{dx} e^{2x} = 2\, e^{2x}$
Using chain rule with $f(x) = e^x$ and $g(x) = 2x$.

(c) $\frac{d}{dx} a^x = \frac{d}{dx}\left(e^{\ln a}\right)^x = \frac{d}{dx} e^{x \ln a} = \ln a\, e^{a \ln x} = \ln a \times a^x$.
So, we have $\frac{d}{dx} 2^x = \ln 2 \times 2^x$ and $\frac{d}{dx} 10^x = \ln 10 \times 10^x$.

In general, we have: $\frac{d}{dx} e^{g(x)} = g'(x)\, e^{g(x)}$.

Natural Log
Recall that logarithms to base e are denoted ln.
To calculate its derivative, we observe that $g(x) = \ln x$ is the inverse function of $f(x) = e^x$.
Thus, a point P with coordinates $(a, b) = (a, e^a)$ on $y = e^x$ is reflected in the line $y = x$ to the point Q with coordinates $(b, a) = (b, \ln b)$ on the curve $y = \ln x$.
Moreover, the tangent in P (to the curve $y = e^x$) has gradient $f'(a) = e^a$, while the tangent in Q (to the curve $y = \ln x$) has gradient

$$\frac{1}{f'(a)} = \frac{1}{e^a} = \frac{1}{e^{\ln b}} = \frac{1}{b}.$$



This shows that the derivative of $\ln x$ is:

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

EXAMPLES:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\ \ln(2\,x+1) \;=\; \frac{2}{2\,x+1}.$
Chain rule with $f(x) = \ln x$ and $g(x) = 2\,x+1$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\ \ln(\tan(x^2+1)) \overset{\text{chain rule}}{=} \frac{1}{\tan(x^2+1)} \times \left(\frac{\mathrm{d}}{\mathrm{d}x}\ \tan(x^2+1)\right)$
$\overset{\text{chain rule}}{=} \frac{1}{\tan(x^2+1)} \times \frac{1}{\cos^2(x^2+1)}\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\,(x^2+1)\right) \;=\; \frac{2x}{\sin(x^2+1)\,\cos(x^2+1)}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\ \log_a x \;=\; \frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\ln x}{\ln a} \;=\; \frac{1}{\ln a}\,\frac{\mathrm{d}}{\mathrm{d}x}\ \ln x \;=\; \frac{1}{x\,\ln a} \qquad \left(= \frac{\log_a \mathrm{e}}{x}\right)$

In general, we have: $\frac{\mathrm{d}}{\mathrm{d}x}\ \ln f(x) \;=\; \frac{f'(x)}{f(x)}.$

Further Examples (not lectured)

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\ \mathrm{e}^{\mathrm{e}^x} \;=\; \mathrm{e}^{\mathrm{e}^x} \times \frac{\mathrm{d}}{\mathrm{d}x}\ \mathrm{e}^x \;=\; \mathrm{e}^{\mathrm{e}^x} \times \mathrm{e}^x \;=\; \mathrm{e}^{\mathrm{e}^x+x}\,.$

Using chain rule with $f(x) = \mathrm{e}^x$ and $g(x) = \mathrm{e}^x$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\ (x\ \mathrm{e}^x) \overset{\text{product rule}}{=} \mathrm{e}^x + x\ \mathrm{e}^x \;=\; (1+x)\ \mathrm{e}^x\,.$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\ (\mathrm{e}^x\ \ln x) \overset{\text{product rule}}{=} \mathrm{e}^x \times \ln x + \mathrm{e}^x \times \frac{1}{x} \;=\; \mathrm{e}^x\ \left(\frac{1}{x} + \ln x\right).$

END OF LECTURE 21

Maxima and Minima

Let y be a function of x. Remember that:

If $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$, then y is *increasing.*

If $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$, then y is *decreasing.*

If $\frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = 0$, then y is *stationary* at x_0. At a stationary point x_0 the gradient is zero, and x_0 may be a minimum or maximum of the function.

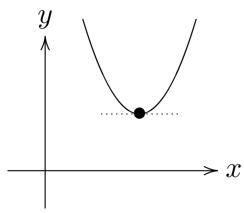
Differentiating y again we get the second derivative: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ (also written $f''(x)$ if $y = f(x)$). As above,

If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ then $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing,

if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ then $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing.

So, if

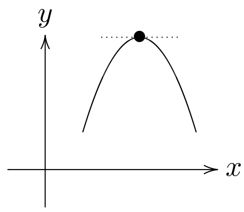
$$\frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = 0 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x_0) > 0 \quad \text{at a point } x_0$$



then the gradient of the tangent is zero but increasing at x_0, so we have a (local) *minimum* at x_0, i.e., x_0 is a minimum point in a region near enough to x_0.

Similarly, if

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = 0 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x_0) < 0 \quad \text{at a point } x_0$$



then we have a (local) *maximum* point on the curve.

NOTE: Stationary points with zero 2nd derivative may be maximum, minimum or neither.

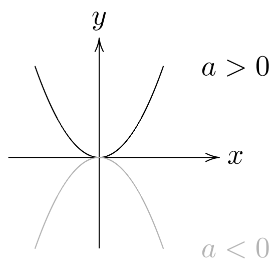
EXAMPLES:

(a) $y = a\,x^2$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2a\,x$, so stationary point at $x_0 = 0$.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2a$.

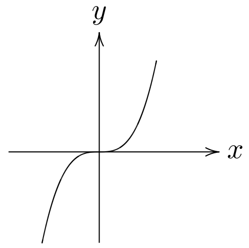
So $x_0 = 0$ is a mimium point if $a > 0$ and maximum point if $a < 0$.



(b) $y = x^3$.

$\frac{dy}{dx} = 3\,x^2, \quad \frac{d^2y}{dx^2} = 6x.$

Both vanish at $x_0 = 0$.
The point $x_0 = 0$ is neither a maximum nor a minimum (it is a so-called *saddle point*).



A Note and Further Examples (examples not lectured)

NOTE: Let y be a function of x.

If $\frac{d^2y}{dx^2} > 0$, then y makes a *left-hand turn.*

If $\frac{d^2y}{dx^2} < 0$, then y makes a *right-hand turn.*

If $\frac{d^2y}{dx^2}(x_0) = 0$, then y has a *point of inflection* at x_0. At an inflection point x_0 the the direction of the bend/turn of the graph may change.

EXAMPLES:

(a) $y = e^{-x^2}$.

$$\frac{dy}{dx} = -2xe^{-x^2};$$

so stationary point for $x_0 = 0$. With

$$\frac{d^2y}{dx^2} = 4x^2e^{-x^2} - 2e^{-x^2}$$

we get $\frac{d^2y}{dx^2}(0) = -2$ and so this function has a maximum at $x = 0$. Note that $e^{-x^2} \to 0$ as $x \to \pm\infty$ (make a sketch of this function!).

(b) Find the greatest and least values of $f(x) = x^3 - 12x$ subject to $-3 \leq x \leq 5$.

Maximising $f(x)$ on an interval $a \leq x \leq b$, the maximum/minimum must be attained at a turning point or an end point.

Calculate turning points: $f'(x) = 3\,x^2 - 12$, thus by $0 = f'(x) = 3\,x^2 - 12$ the turning points are at $x_1 = -2$ and $x_2 = 2$. With $f''(x) = 6\,x$ get that at $x_1 = -2$ we have a local maximum and at $x_2 = 2$ a local minimum.

Values at turning points and end points: $f(-3) = 9$, $f(-2) = 16$, $f(2) = -16$ and $f(5) = 65$. Thus, the greatest value of $f(x)$ on the interval $-3 \leq x \leq 5$ is attained at $x = 5$ (where $f(x) = 65$) and the least value is attained at $x = 2$ (where $f(2) = -16$).
(Make a sketch of $y = f(x)$.)

Uses of Differentiation

EXAMPLE: A circular colony of bacteria is increasing in radius at 3 mm/hr. If its current radius is 2 cm, find the current rate of increase of its area.

$A = \pi r^2$, where $r = r(t)$.
So, by chain rule,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) \times \frac{\mathrm{d}r}{\mathrm{d}t} = 2\pi\, r\, \frac{\mathrm{d}r}{\mathrm{d}t}$$

At the current time $r = 2\,\text{cm}$ and $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{10}\,\text{cm/hr}$.
So, substituting into the above equation, we obtain

$$\frac{dA}{dt} = 2\pi \times 2\,\text{cm} \times \frac{3}{10}\,\text{cm/hr} = \frac{6\,\pi}{5}\,\text{cm}^2/\text{hr}.$$

EXAMPLE: A particle moves so that at time t it is situated at the point $P = (t-4, 2+t)$. Find the closest the particle gets to the origin $(0,0)$, and at what time this occurs.

For example, at the initial time $t = 0$ we have $P = (-4, 2)$, while at $t = 1$ it has moved to $P = (-3, 3)$ (in fact, it is a straight line!).
At any time t, square of the distance of P from $(0,0)$: $f(t) = (t-4)^2 + (2+t)^2 = 2t^2 - 4t + 20$. Thus $f'(t) = 4t - 4$ so the only stationary point is $t = 1$. Since $f''(t) = 4$ for all t, the point $t = 1$ is a (local) minimum.
Alternatively (instead of using the second derivative), we can argue that since $f'(t)$ is positive for $t > 1$ and negative for $t < 1$, the smallest value of $f(t)$ must occur at $t = 1$. So the minimal distance of the particle from $(0,0)$ is $\sqrt{f(1)} = \sqrt{18}$.

Another example (not lectured)

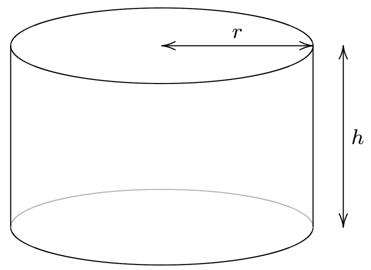
Repeat the previous question, but for the point $P = (\cos t, 2\sin t)$.

Here $f(t) = \cos^2 t + 4\sin^2 t$.
Hence $f'(t) = -2\cos t \sin t + 8 \sin t \cos t = 6 \sin t\, \cos t$.
Stationary points satisfy either $\sin t = 0$ or $\cos t = 0$, so $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$. By the periodicity of sin and cos we only have to check what happens at $t = 0,\ \pi/2,\ \pi,\ 3\pi/2$.

$$\begin{aligned} f''(t) &= 6\cos^2 t - 6\sin^2 t \\ &= \begin{cases} 6 & \text{at } 0,\ \pi \quad \text{(minima)} \\ -6 & \text{at } \frac{\pi}{2},\ \frac{3\pi}{2} \quad \text{(maxima)} \end{cases} \end{aligned}$$

Since $f(0) = f(\pi) = 1$, the minimum value of $\sqrt{f(t)}$ is $\sqrt{1} = 1$.

EXAMPLE: A cylindrical can with an open top is to be made to contain unit volume; find the dimensions that minimise the area of metal needed.



Volume of can $V = \pi r^2 h = 1$ so $h = \frac{1}{\pi r^2}$
Thus

$$\text{Area } A = 2\pi r h + \pi r^2 = 2\pi r \frac{1}{\pi r^2} + \pi r^2 = \frac{2}{r} + \pi r^2$$

Hence $\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{2}{r^2} + 2\pi r$ and this vanishes when $2\pi r = 2/r^2$, i.e., $r = (1/\pi)^{1/3}$. Since $\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = \frac{4}{r^3} + 2\pi > 0$ for positive r, the point $r = (1/\pi)^{1/3}$ is a local minimum point. Since A increases to ∞ as $r \to \infty$, this is the minimum over all r.
The corresponding height h is $\left(\frac{1}{\pi}\right)^{1/3}$

END OF LECTURE 22

Merry Christmas and A Happy New Year – Good Luck in the Final Exam

I hope that I was able to explain some of the basic ideas and concepts in calculus and you learned and understood something that will prove useful in your sciences classes (either in chemistry, natural sciences or sport & exercise).

Good luck with your further studies!