MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 10

Derivatives of $\sin$ **and** $\cos$

To calculate the derivative of $\sin x$, we must find the following limit as $\delta x \to 0$:

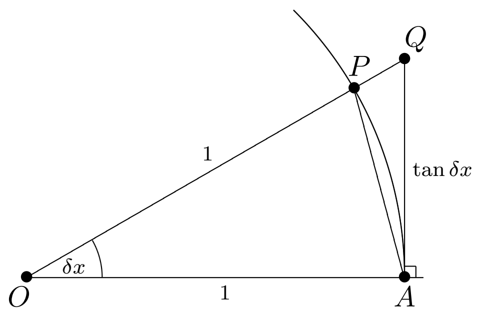
$$\frac{\sin(x+\delta x)-\sin x}{\delta x} = \frac{\sin x \times \cos\delta x + \cos x \times \sin\delta x - \sin x}{\delta x}$$
$$= \sin x \times \left(\frac{\cos\delta x \, - \, 1}{\delta x}\right) + \cos x \times \left(\frac{\sin\delta x}{\delta x}\right) \qquad (1)$$

As $\delta x \to 0$, what do we get for the expressions in the brackets?

(The following derivation is not examinable!)

In the picture on the right,

- the area of the triangle OAP is given by $\frac{1}{2} \times 1 \times 1 \times \sin\delta x$,
- the area of the sector OAP is $\frac{\delta x}{2\pi}$ times the area of the full circle and therefore $\frac{\delta x}{2\pi} \times \pi \times 1^2 = \frac{1}{2}\delta x$,
- and the area of the triangle OAQ is $\frac{1}{2} \times 1 \times \tan\delta x$.



We have:

$$\begin{array}{ccccccc} \text{area } \Delta\, OAP & \leq & \text{area of sector } OAP & \leq & \text{area } \Delta\, OAQ & & \\ \frac{1}{2}\sin\delta x & \leq & \frac{1}{2}\,\delta x & \leq & \frac{1}{2}\tan\delta x & = & \frac{1}{2} \times \frac{\sin\delta x}{\cos\delta x} \end{array}$$

from which we get the inequalities $\sin\delta x \leq \delta x$ (rewritten $\frac{\sin\delta x}{\delta x} \leq 1$) and $\delta x \leq \frac{\sin\delta x}{\cos\delta x}$ (rewritten $\cos\delta x \leq \frac{\sin\delta x}{\delta x}$). Therefore,

$$\cos\delta x \leq \frac{\sin\delta x}{\delta x} \leq 1,$$

and, as $\delta x \to 0$, we have $\cos\delta x \to 1$ ans so also $\frac{\sin\delta x}{\delta x} \to 1$.
Similarly, one can show that $\frac{\cos\delta x - 1}{\delta x} \to 0$ as $\delta x \to 0$.

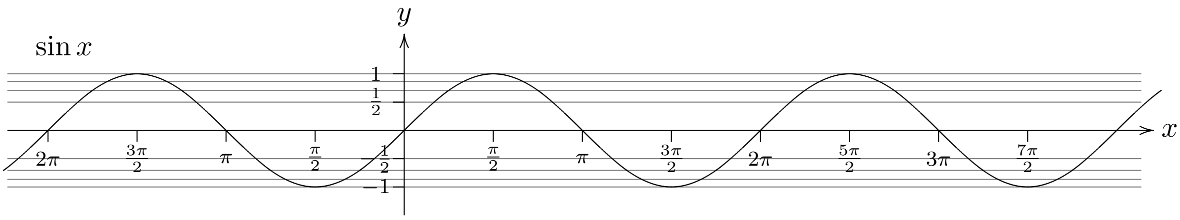
Inserting these into (1) yields:

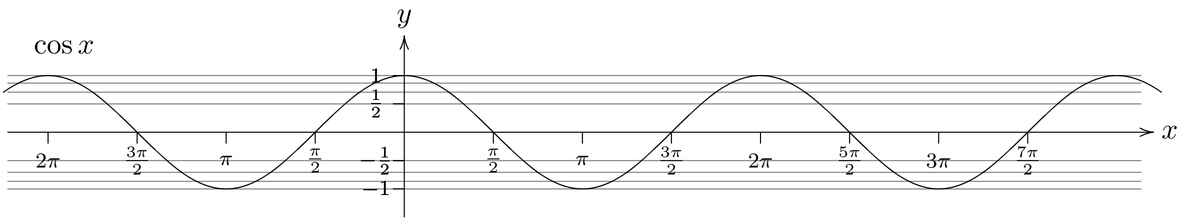
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x}$$

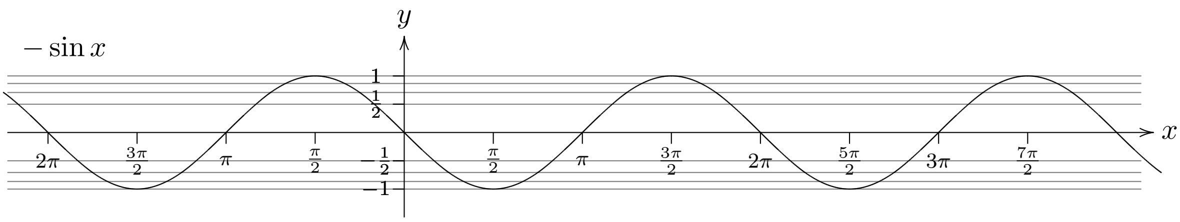
Similarly, it can be shown that

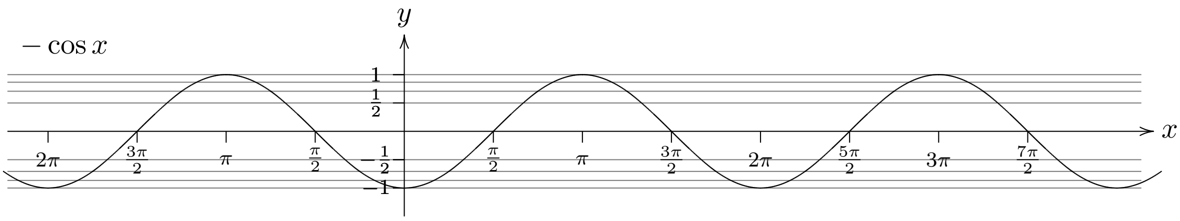
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x}$$

Warning: **These formulas hold if the angle x is measured in radians!**









Differentiating more complicated functions

To do more complicated derivatives we need three more rules.

Product Rule

If f and g are both functions of x, we can differentiate the product of f and g as follows:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\,g(x)\right) \;=\; \left(\tfrac{\mathrm{d}}{\mathrm{d}x}\,f(x)\right)\,g(x) \;+\; f(x)\,\left(\tfrac{\mathrm{d}}{\mathrm{d}x}\,g(x)\right) \;=\; f'(x)\,g(x) \;+\; f(x)\,g'(x)}$$

Proof of the product rule (not lectured, not examinable)

To proof the product rule, we make the following calculation:

$$\begin{aligned}
&\frac{f(x+\delta x)\,g(x+\delta x) \;-\; f(x)\,g(x)}{\delta x} \\
&\quad= \frac{f(x+\delta x)\,g(x+\delta x) \;-\; f(x)\,g(x+\delta x) \;+\; f(x)\,g(x+\delta x) \;-\; f(x)\,g(x)}{\delta x} \\
&\qquad= \frac{f(x+\delta x)-f(x)}{\delta x} \times g(x+\delta x) \;+\; f(x) \times \frac{g(x+\delta x)-g(x)}{\delta x} \\
&\qquad\qquad \to\; f'(x)\,g(x) \;+\; f(x)\,g'(x) \qquad \text{as} \quad \delta x \to 0
\end{aligned}$$

EXAMPLES:

(a) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left((2x+1)(x+1)\right) \;=\; 2\times(x+1)+(2x+1)\times 1 \;=\; 4x+3.$

Here we have put $f(x)=(2x+1)$ and $g(x)=(x+1)$; thus $f'(x)=2$ and $g'(x)=1$.

(b) $$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}x}\left((2x^2+3)^2\right) &= (2x^2+3)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\,(2x^2+3)+(2x^2+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\,(2x^2+3) \\ &= 2\,(2x^2+3)\,4\,x = 16\,x^3 \;+\; 24\,x.\end{aligned}$$

Here we have put $f(x)=g(x)=(2x^2+3)$ wherefore $f'(x)=g'(x)=4\,x$).

Quotient Rule

If f and g are both functions of x, we can differentiate the quotient of f and g as follows:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right) g(x) - f(x)\left(\frac{\mathrm{d}}{\mathrm{d}x} g(x)\right)}{(g(x))^2} = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{(g(x))^2}}$$

Special case $f(x) = 1$: In this case, since $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = 0$, quotient rule gives:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(x)}\right) = -\frac{\frac{\mathrm{d}}{\mathrm{d}x} g(x)}{(g(x))^2} = -\frac{g'(x)}{(g(x))^2}}$$

EXAMPLES:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x+2}\right) = -\frac{1}{(x+2)^2}.$

Here $f(x) = 1$ and $g(x) = (x+2)$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2+2}\right) = -\frac{2\,x}{(x^2+2)^2}.$

Here $f(x) = 1$ and $g(x) = x^2 + 2$ (und therefore $g'(x) = 2\,x$).

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x+1}\right) = \frac{1 \times (x+1) - x \times 1}{(x+1)^2} = \frac{1}{(x+1)^2}.$

Here $f(x) = x$, $g(x) = (x+1)$.

(d) $\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$

Here $f(x) = \sin x$ and $g(x) = \cos x$.

Another possibility is $\frac{\mathrm{d}}{\mathrm{d}x} \tan x = 1 + \tan^2 x$.

We have shown

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \frac{1}{\cos^2 x}}$$

Chain Rule

Suppose that y can be written as a function of g, where g is a function of x. An example would be $y = (x+1)^3$, which can be written as $y = g(x)^3$ where $g(x) = (x+1)$. Hence, the complicated function $y = (x+1)^3$ is written in terms of two simpler functions: $y = (x+1)^3 = f(x+1) = f(g(x))$ where $f(x) = x^3$ and $g(x) = x+1$. The notation $f(g(x)) = (f \circ g)(x)$ is also used ("$\circ$" is the "chain joint").

Suppose that as x increases by δx, then g increases by δg and $y = f \circ g$ increases by δy. Then we are tempted to write:

$$\frac{\delta y}{\delta x} = \frac{\delta (f \circ g)}{\delta x} \overset{\substack{f \text{ is function of } g, \\ g \text{ is function of } x}}{=} \frac{\delta f}{\delta g} \times \frac{\delta g}{\delta x} = \frac{\delta y}{\delta g} \times \frac{\delta g}{\delta x}.$$

Compare this with the following "stupid" example: $y = 6\,x = 3\,(2\,x) = f(g(x))$ with $f(x) = 3\,x$ and $g(x) = 2\,x$ (therefore, $f(g(x)) = 3\,g(x) = 3 \times 2\,x = 6\,x$ and $f'(x) = 3$ and $g'(x) = 2$). Here, of course, we have $\frac{\mathrm{d}x}{\mathrm{d}y} = 6 = 3 \times 2$.

Indeed, these considerations are the correct idea, and one obtains the *chain rule*:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} f\,(g(x)) = f'(g(x)) \times g'(x)}$$

END OF LECTURE 19

If y is a function of some function $g(x)$, i.e., $y = f(g(x))$, its derivative can be calculated using the *chain rule*:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} f\left(g(x)\right) \;=\; f'(g(x))\, g'(x)}$$

EXAMPLES:

(a) $$\frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)^3\right) \overset{f(x)=x^3,\; g(x)=x+1}{=} 3\,(x+1)^2 \times 1 \;=\; 3\,(x+1)^2.$$

Here, $f(x) = x^3$ and $g(x) = x+1$. Check that $f(g(x)) = (g(x))^3 = (x+1)^3$ and note that $f'(x) = 3x^2$ and $g'(x) = 1$.

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(x+2)^2}\right) \overset{f(x)=x^{-2},\; g(x)=x+2}{=} -2\,(x+2)^{-3} \times 1 \;=\; -\frac{2}{(x+2)^3}$$

Here, $f(x) = \frac{1}{x^2} = x^{-2}$ and $g(x) = x+2$. Check that $f(g(x)) = (g(x))^{-2} = (x+2)^{-2} = \frac{1}{(x+2)^2}$ and note that $f'(x) = -2x^{-3}$ and $g'(x) = 1$.

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(2x) = 2\,\cos(2x)$.

Here, we use the chain rule with $f(x) = \sin x$ and $g(x) = 2x$.

(d) $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(1+x)^7}{(1-x)^5}\right) &\overset{\text{quotient rule}}{=} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}(1+x)^7\right)(1-x)^5 \;-\; (1+x)^7\left(\frac{\mathrm{d}}{\mathrm{d}x}(1-x)^5\right)}{\left((1-x)^5\right)^2} \\ &\overset{\text{chain rule}}{=} \frac{7\,(1+x)^6 \times 1 \times (1-x)^5 \;-\; (1+x)^7 \times 5 \times (1-x)^4 \times (-1)}{(1-x)^{10}} \\ &= \frac{(1-x)^4(1+x)^6\;\left(7\,(1-x) + 5\,(1+x)\right)}{(1-x)^{10}} \\ &= \frac{(1+x)^6}{(1-x)^6}\,(12-2x)\,.\end{aligned}$$

Inverse functions

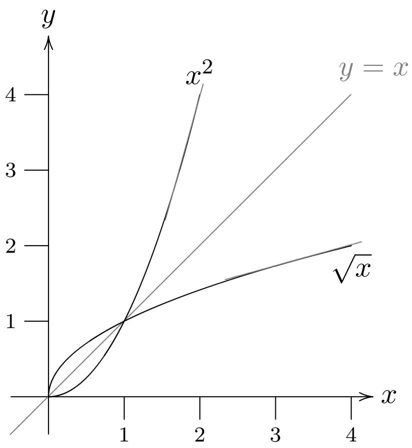
Recall: If $y^2 = x$, then y is not a function of x, because for $x > 0$ there are two possible values of y, namely $\sqrt{x}$ and $-\sqrt{x}$.
On the other hand, the formula $y = \sqrt{x}$ defines y uniquely as the positive square root of x and in this case y is a function of x.
Also note that this function is only defined for $x \geq 0$.

Generally, the set of x-values where a function is defined is called its *domain*. For a given domain and function f, the set of "output numbers" is called the *range* of the function.

EXAMPLES: The range of $f(x) = x^2$ (with domain $\mathbb{R}$) is the set of all nonnegative numbers (i.e., all positive numbers and 0). Negative numbers are not in the domain of $\sqrt{x}$.

For $x \geq 0$, we set $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then, we have $f(g(x)) = (\sqrt{x})^2 = x$ and $g(f(x)) = \sqrt{x^2} = x$, i.e., on the domain of nonnegative numbers, $f(x) = x^2$ and $g(x) = \sqrt{x}$ are *inverse* to each other (e.g., using the output of f as input in g yields the original input of f).

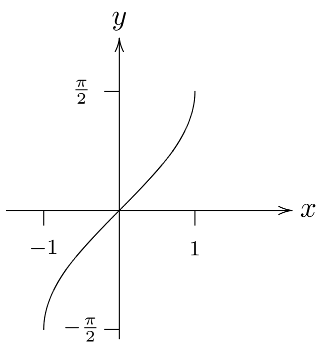
If we consider the curve $y = f(x)$, then the curve $y = g(x)$ of its inverse is obtained by reflecting in the line $y = x$.



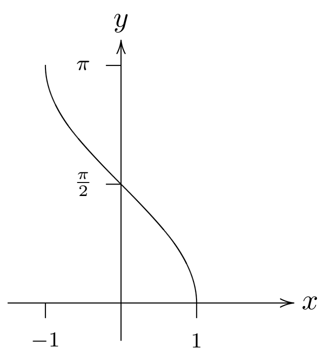
Inverse trigonometric functions

Since we have for the range of sine and cosine that $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$, the domain of their inverse functions is the interval from -1 to 1. The inverse function of *sinx* is called *arcsine* (its range is usually chosen to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$) and denoted $\arcsin x$, the inverse function of cosine is *arccosine* (its range is usually chosen to be between 0 and π) and denoted $\arccos x$.

The graph of $\arcsin x$:



The graph of $\arccos x$:



Similarly, the inverse function of $\tan x$ is called *arctangent* and denoted $\arctan x$.

NOTE: The notation $\sin^{-1} x$ (e.g., on the calculator) for the inverse function of sine is misleading: If we use the notation $\sin^2 x = (\sin x)^2$, then $\sin^{-1} x$ should be $(\sin x)^{-1} = \frac{1}{\sin x}$. But this is clearly not $\arcsin x$.

END OF LECTURE 20