MA10103: Foundation Mathematics I

LECTURE NOTES – WEEK 1

§1 Numbers

The *naturals* are the "nonnegative whole numbers", i.e., $0, 1, 2, 3, 4, \ldots$. The set of naturals is denoted by $\mathbb{N}$.

Warning: Sometimes only the positive integers (i.e., $1, 2, 3, 4, \ldots$) are called natural numbers

The *integers* are the "whole numbers" i.e. $\ldots, -5, -4, \cdots, -1, 0, 1, 2, 3, \ldots$. The set of integers is denoted by $\mathbb{Z}$.

The *rationals* are the "fractions", e.g., $\frac{1}{2}, -\frac{2}{5}, \frac{11}{3}$ etc. In general,

$$\text{Fraction} = \frac{\text{numerator}}{\text{denominator}}$$

where *numerator* and a *denominator* are both integers (the denominator is never 0 and often chosen > 0). The set of rationals is denoted by $\mathbb{Q}$.

We simplify by cancelling any common factors, e.g., $\frac{3}{6} = \frac{1}{2}$.

All integers are rational, e.g., $3 = \frac{3}{1}$.

Note: Adding/multiplying naturals/integers/rationals yields a natural/integer/rational; substracting integers/rationals yields an integer/rational; dividing rationals yields a rational.

Multiplying Fractions

$$\text{e.g.:} \qquad \frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

Dividing by Fractions

"Turn divisor upside down then multiply"

$$\text{e.g.:} \qquad \frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15} \qquad \text{or} \qquad \frac{2}{3} \div \frac{5}{7} = \frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{\cancel{3}} \times \cancel{3} \times 7}{\frac{5}{\cancel{7}} \times 3 \times \cancel{7}} = \frac{2 \times 7}{5 \times 3} = \frac{14}{15}$$

Adding/subtracting fractions

Easy case, e.g.: $\frac{1}{5} + \frac{3}{5} = \frac{1+3}{5} = \frac{4}{5}$

Less easy, e.g.: $\frac{1}{5} + \frac{2}{3}$

To do this, make a "*common denominator*":

$$\frac{1}{5} + \frac{2}{3} \overset{\text{multiply by 1}}{=} \left(\frac{1}{5} \times \frac{3}{3}\right) + \left(\frac{2}{3} \times \frac{5}{5}\right) = \frac{3}{15} + \frac{10}{15} = \frac{13}{15}$$

Worked Examples (not given in the lecture)

$$\frac{3}{7}-\frac{1}{8} = \frac{3\times 8}{7\times 8}-\frac{1\times 7}{8\times 7} = \frac{24}{56}-\frac{7}{56} = \frac{17}{56}$$

$$\frac{2}{9}+\frac{5}{18} \overset{18=2\times 9}{=} \frac{2\times 2}{9\times 2}=\frac{5}{18} = \frac{4}{18}+\frac{5}{18} = \frac{9}{18} = \frac{1}{2}$$

Real numbers

This is the set of all possible numbers, containing integers and rationals, but also numbers like $\sqrt{2}$, π and $\sqrt{7}$. The set of real numbers is denoted by $\mathbb{R}$.

We may show next week that $\sqrt{2}$ is not a rational number.

We can write all reals as decimals, e.g.,

$$\begin{aligned}
\frac{1}{4} &= 0.25 \text{ (terminates)}\\
\frac{1}{3} &= 0.3333\ldots = 0.\overline{3} \text{ (recurs)}\\
\sqrt{2} &= 1.414213\ldots \text{ (neither terminates nor recurs)}\\
\frac{4}{33} &= 0.121212\ldots = 0.\overline{12} \text{ (recurs)}
\end{aligned}$$

The line above digits indicate that they recur.

Note: Decimal digits of rational numbers either terminate or recur. Decimal digits of irrational numbers neither terminate nor recur.

On the "all possible numbers" (neither given in lecture and nor examinable)

What does "all possible" mean? Look at the following sequence of rational numbers: $1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \ldots$. The $(n+1)$th number x_{n+1} in this sequence is calculated from its predecessor, the nth number x_n, by

$$x_{n+1} = \frac{x_n^2+2}{2\,x_n}.$$

E.g., using $x_1 = \frac{3}{2}$ one calculates $(\frac{3}{2}\times\frac{3}{2}+2)/(2\times\frac{3}{2}) = (\frac{17}{4})/3 = \frac{17}{12} = x_2$.
Now, look at the following table:

n	x_n	x_n (6 d.p.)	x_n^2	x_n^2 (6 d.p.)
0	1	1.000	1	1.000
1	$\frac{3}{2}$	1.500	$\frac{9}{4}$	2.250
2	$\frac{17}{12}$	1.416667	$2\,\frac{1}{144}$	2.006944
3	$\frac{577}{408}$	1.414216	$2\,\frac{1}{166464}$	2.000006

The squares x_n^2 approach 2, and in fact, the sequence of rational numbers x_n "converges to the limit" $\sqrt{2}$. "All possible" means that each real number can be obtained as limit of a converging sequence of rational numbers. Note, that the decimal expansion itself yields a converging sequence of rational numbers, e.g., $1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \ldots$ also converges to $\sqrt{2}$. Also note that the limit of a converging sequence of real numbers is again a real number.

Next week, we might show that $\sqrt{2}$ is irrational, i.e., a real number that is not rational.

We may approximate decimals to a number of decimal places (d.p.) or significant figures (s.f.).

Decimal places

$$\sqrt{2} = 1.\underset{\substack{\uparrow\\ 1^{\text{st}}\text{d.p.}}}{4}1\,4\,2\,1\,3\,\underset{\substack{\uparrow\\ 7^{\text{th}}\text{d.p.}}}{5}\,6\,2\cdots$$

so $\sqrt{2} = 1.4$ (1 d.p.), $\sqrt{2} = 1.4142136$ (7 d.p.)
The *first decimal place* is the first digit after the decimal point, etc. (If the first digit not written is ≥ 5 we round the previous digit up).

Significant figures

$$\sqrt{2} = \underset{\substack{\uparrow\\ 1^{\text{st}}\text{ s.f.}}}{1}.4\,1\,4\,\underset{\substack{\uparrow\\ 5^{\text{th}}\text{ s.f.}}}{2}\,1\,3\,5\,6\,2\cdots$$

so $\sqrt{2} = 1$ (1 s.f.), $\sqrt{2} = 1.4142136$ (8 s.f.)
The *first significant figure* is the first non-zero digit, regardless of the position of the decimal point.

EXAMPLES: $999 = 1000$ (1 s.f.); $999 = 999.0$ (1 d.p.)

END OF LECTURE 1

Standard Index form

The gravitational constant G is $0.\underbrace{0\ldots\ldots\ldots0}_{10\text{ zeros}}6673\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$.

It is very inconvenient to write like this, so we use S.I. form : $\mathrm{G} = 6.673\times10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$

The general form is $a \times 10^b$ where $1 \le a < 10$ and b is an integer.

EXAMPLES: $0.003 = 3 \times 10^{-3}$

$1000 = 1 \times 10^3$

$999 = 9.99 \times 10^2$; $999 = 1 \times 10^3$ (1 s.f.)

More Examples

1. Write the speed of light correct to 3 s.f. $c = 300000000\,\mathrm{m\,s^{-1}}$ (3 s.f.)
2. Write the speed of light correct to 5 s.f. in S.I. form. $c = 2.9979 \times 10^8\,\mathrm{m\,s^{-1}}$ (5 s.f.)
3. Write Avogadro's constant correct to 1 s.f. in S.I. form. $N_A = 6.0\times10^{23}\,\mathrm{mol^{-1}}$ (1 s.f.)
4. Write π correct to 2 d.p. in S.I. form. $\pi = 3.14$ (2 d.p.)

Speed of light $c = 2.99792458 \times 10^8\,\mathrm{m\,s^{-1}}$; Avogadro's Constant $N_A = 6.02214 \times 10^{23}\,\mathrm{mol^{-1}}$

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§2 Basic Algebra

Multiplication: $3x$ means 3 multiplied by x.

Use of brackets: $3p^2$ means $3 \times p \times p$, while $(3p)^2 = 3p \times 3p = 9p^2$.

EXAMPLE: $(2a)^3 = 2a \times 2a \times 2a = 2 \times 2 \times 2 \times a \times a \times a = 8a^3$

Division

EXAMPLES:

$$\frac{x}{x} = 1$$

$$\frac{3x^2}{x} = \frac{3 \times x \times x}{x} = 3x$$

$$\frac{3x^2}{y} \div \frac{x}{y} = \frac{3x^2}{y} \times \frac{y}{x} = 3x$$

Of course, we are assuming here that all denominators, e.g., x in the first example, are not zero, because otherwise the left-hand sides would not be defined. To be more precise, the first example should actually read: $\frac{x}{x}$ equals 1 for all x except for $x = 0$ for which $\frac{x}{x}$ is not defined.

Addition/Subtraction

EXAMPLE:

$$\underset{\underset{1^{st}\text{ term}}{\uparrow}}{3x^2y} \;+\; 5xy \;+\; \underset{\underset{3^{rd}\text{ term}}{\uparrow}}{y^3} + 5x^2y$$

Like terms contain the same combination of letters, e.g. $3x^2y$ and $5x^2y$ and can be combined.

Unlike terms cannot be combined

EXAMPLES:

(a) $\underline{3x^2y} + 5xy + y^3 + \underline{5x^2y} = 8x^2y + 5xy + y^3$

(b) $5p^2q - (2p)^2q + 3pq^2 = \underline{5p^2q - 4p^2q} + 3pq^2 = p^2q + 3pq^2$

We say that $3pq^2$ "is the pq^2 term" in (b). Its *coefficient* is 3

Expanding brackets:
$(a+b)x = ax + bx, \quad x(a+b) = xa + xb = ax + bx \quad$ (same as $(a+b)x$)

$(a+b)(p+q) = a(p+q) + b(p+q) = ap + aq + bp + bq$
(Leave out the middle step if you like.)

More Examples

$$\begin{aligned} 3(x-2) &= 3x - 6 \\ (x-1)(y-2) &= xy - y - 2x + 2 \\ (x-a)(x+a) &= x^2 - ax + ax - a^2 \;=\; x^2 - a^2 \end{aligned}$$

A very important formula: $\boxed{(x-a)(x+a) = x^2 - a^2}$
The left-hand side is the factorisation of "the difference of two squares".

Also: $\boxed{(x+a)^2 = x^2 + 2a + a^2}$

and $\boxed{(x-a)^2 = x^2 - 2a + a^2}$

More generally, for any a and b,

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

We can use this to factorise a quadratic expression (i.e., one involving x and x^2) into linear factors (invloving only x).

EXAMPLE: Factorise $x^2 + 5x + 6$.
Here $ab = 6$ and $a + b = 5$. From the latter we deduce $b = 5 - a$ and obtain the following table:

a	$b = 5 - a$	$a\,b$
-1	6	-6
0	5	0
1	4	4
2	3	6
3	2	6
4	1	4
5	0	0
6	-1	-6

Therefore, we obtain $a = 3$ and $b = 2$ (or vice versa) and have the following factorisation:

$$x^2 + 5x + 6 = (x + 3)(x + 2).$$

You don't have to do such a table!

We also note the following:

- The product of two positive numbers is positive. The sum of two positive numbers is positive.
- The product of two negative numbers is positive. The sum of two negative numbers is negative.
- The product of a positive and a negative number is negative.

With this, we can analyse the following examples.

EXAMPLES: Factorise $x^2 + 7x + 12$.
Here, we have $ab = 12$ and $a + b = 7$, thus both a and b are positive.
Indeed, one obtains $x^2 + 7x + 12 = (x + 3)(x + 4)$.

Factorise $x^2 + 4x - 12$.
Here, we have $ab = -12$ and $a + b = 4$, thus one of the two numbers is positive the other negative.
One obtains $x^2 + 4x - 12 = (x - 2)(x + 6)$.

END OF LECTURE 2