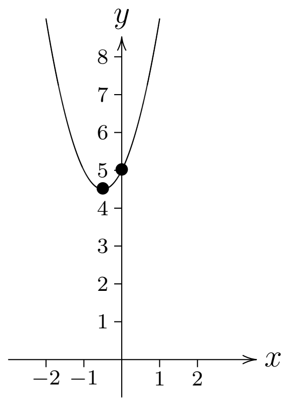
MA10103: Foundation Mathematics I

Solutions of Problem Sheet 9 (assessed coursework)

1. Use $\cos^2 A = 1 - \sin^2 A$ (respectively $\sin^2 A = 1 - \cos^2 A$) in double angle identity to get
$$\cos(2A) \overset{[1]}{=} 2\cos^2 A - 1 \quad \text{and} \quad \cos(2A) \overset{[1]}{=} 1 - 2\sin^2 A.$$
Thus,
$$\cos A \overset{[\frac{1}{2}]}{=} \pm\sqrt{\frac{1+\cos(2A)}{2}} \quad \text{and} \quad \sin A \overset{[\frac{1}{2}]}{=} \pm\sqrt{\frac{1-\cos(2A)}{2}}.$$
With $A = 22.5°$, we have the plus-sign in both cases [1] and obtain
$$\sin 22.5° \overset{[\frac{1}{2}]}{=} \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \overset{[\frac{1}{2}]}{=} \frac{\sqrt{2-\sqrt{2}}}{2} \overset{[\frac{1}{2}]}{=} 0.383 \text{ (3 d.p.)} \quad \text{and}$$
$$\cos 22.5° \overset{[\frac{1}{2}]}{=} \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} \overset{[\frac{1}{2}]}{=} \frac{\sqrt{2+\sqrt{2}}}{2} \overset{[\frac{1}{2}]}{=} 0.924 \text{ (3 d.p.)}.$$
Or simply have $A = 22.5°$ and $2A = 45°$ in all calculations above.

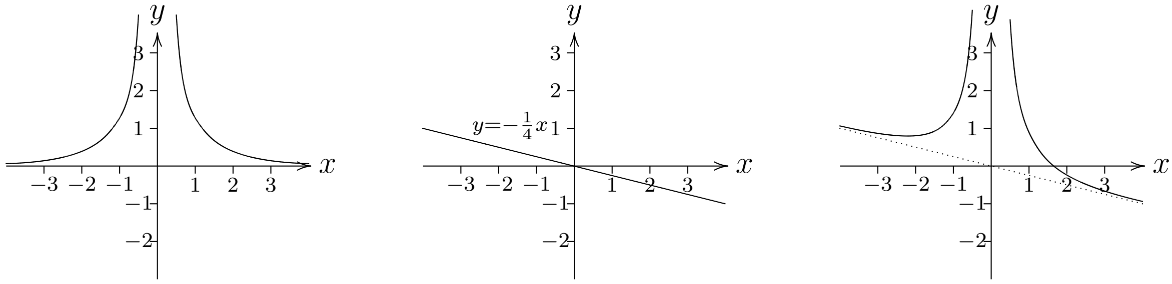
2. Complete the square:
$f(x) = 2x^2 + 2x + 5 = 2\left(x^2 + x\right) + 5 = 2\left(x^2 + x + \frac{1}{4}\right) + 5 - 2 \times \frac{1}{4} = 2\left(x + \frac{1}{2}\right)^2 + \frac{9}{2}$. [2]
Therefore, lowest point at $(-\frac{1}{2}, \frac{9}{2})$. [1]
In particular, this function is always positive and thus has no x-intercepts. [1]
y-intercept: $(0, 5)$. [1]
Sketch [1]: see figure on the right.



3. Equation of circle: $(x+2)^2 + y^2 = 4^2$ [1]. Substituting equation of line into circle yields
$$16 = (x+2)^2 + (x-1)^2 \overset{[1]}{=} 2x^2 + 2x + 5.$$
Solution of the quadratic equation $2x^2 + 2x - 11 = 0$:
$$x \overset{[\frac{1}{2}]}{=} \frac{-2 \pm \sqrt{4+88}}{4} \overset{[\frac{1}{2}]}{=} -\frac{1}{2} \pm \frac{\sqrt{23}}{2}.$$
So, points of intersection are $(-\frac{1}{2} - \frac{\sqrt{23}}{2}, -\frac{3}{2} - \frac{\sqrt{23}}{2})$ and $(-\frac{1}{2} + \frac{\sqrt{23}}{2}, -\frac{3}{2} + \frac{\sqrt{23}}{2})$. [1]

Please turn over!

4. – Sketch of $y = \frac{1}{x^2}$: asymptotes $x = 0$ and $y = 0$ [1], symmetrical (even function) & positive [1], shape [1]
   – Sketch of $y = -\frac{1}{4}x$: straight line (odd function) [1]
   – Sketch of $y = -\frac{1}{4}x + \frac{1}{x^2}$: asymptotes $x = 0$ and $y = -\frac{1}{4}x$ [1], x-intercept at $(\sqrt[3]{4}, 0) \approx (1.587, 0)$ [1], shape (e.g., local minimum at $(-2, \frac{3}{4})$) [2]



5. Derivative of $f(x) = 3x^2 - 1$ is $f'(x) = 6x$ [1], so the gradient of the tangent at $(1, 2)$ is $f'(1) = 6$ [1].
   Equation of tangent [1]: $y = 6x - 4$.

6. (a) With $y = x^3 + x^5$, we get $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 5x^4$ [2].
   (b) With $y = x^{1/5} + x^7 + x^{-2}$ [1], we get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{5}\, x^{-4/5} + 7\, x^6 - 2\, x^{-3}$ [3].

7. P has coordinates $(1, 0)$ [$\frac{1}{2}$].
   Derivative $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{1}{3}\, x^{-4/3}$ [1], so gradient of tangent at P is $\frac{4}{3}$ [1] and that of normal is $-\frac{3}{4}$ [$\frac{1}{2}$].
   Equation of tangent is $y = \frac{4}{3}x - \frac{4}{3}$ [1]; equation of normal is $y = -\frac{3}{4}x + \frac{3}{4}$ [1].

8. Using abbreviations R (Royal Crescent), C (Circus), T (Theatre) and G (Guildhall).
   Angle at R ($\sphericalangle TRC$) is 45°, angle at G ($\sphericalangle TGC$) is 50°. [1]
   Use sine-rule to calculate distances $|RC|$ and $|CG|$ (or, alternatively, $|RT|$ and $|TG|$): [1]

$$|RC| \overset{[1]}{=} 440\,\mathrm{m} \times \frac{\sin 20°}{\sin 45°} = 212.82\,\mathrm{m}\ (2\text{ d.p.}) \quad \text{and}$$

$$|CG| \overset{[1]}{=} 440\,\mathrm{m} \times \frac{\sin 95°}{\sin 50°} = 572.19\,\mathrm{m}\ (2\text{ d.p.})$$

Please turn over!

Use cosine-rule to calculate $|RG|$ via

$$|RG|^2 \overset{[1]}{=} |RC|^2 + |CG|^2 - 2\,|RC|\,|CG|\,\cos(115^\circ + 35^\circ),$$

which yields $|RG| = 764\,\mathrm{m}$ (0 d.p.). [1]

(alternatively,

$$|RT| = 440\,\mathrm{m} \times \frac{\sin 115^\circ}{\sin 45^\circ} = 563.95\,\mathrm{m}\ (2\ \text{d.p.}) \quad \text{and}$$

$$|TG| = 440\,\mathrm{m} \times \frac{\sin 35^\circ}{\sin 50^\circ} = 329.45\,\mathrm{m}\ (2\ \text{d.p.})$$

then $|RG|$ via

$$|RG|^2 = |RT|^2 + |TG|^2 - 2\,|RT|\,|TG|\,\cos(95^\circ + 20^\circ),$$

which again yields $|RG| = 764\,\mathrm{m}$ (0 d.p.).)

Total: [45]