MA10103: Foundation Mathematics I

SOLUTIONS OF PROBLEM SHEET 8

1. $\sin 15° = \sin(45° - 30°) = (\sin 45°) \times (\cos 30°) - (\cos 45°) \times (\sin 30°) = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$; $\cos 15° = (\cos 45°) \times \cos(30°) + (\sin 45°) \times (\sin 30°) = \frac{\sqrt{6}+\sqrt{2}}{4}$; $\sin 75° = (\sin 45°) \times (\cos 30°) + (\cos 45°) \times (\sin 30°) = \frac{\sqrt{6}+\sqrt{2}}{4}$; $\cos 315° = \cos(360° - 45°) = \cos 45° = \frac{\sqrt{2}}{2}$; $\tan 75° = \frac{\sin 75°}{\cos 75°} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{8+2\sqrt{12}}{4} = 2 + \sqrt{3}$; $\sin 135° = \sin(90° + 45°) = \cos(45°) = \frac{\sqrt{2}}{2}$; $\tan 105° = \frac{\sin(90°+15°)}{\cos(90°+15°)} = \frac{\cos 15°}{-\sin 15°} = -\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = -2 - \sqrt{3}$; $\sin^2 30° + \cos^2 30° = 1$; $\tan^2 45° - \frac{1}{\cos^2 45°} = \frac{(\sin^2 45°)-1}{\cos^2 45°} \overset{1=\cos^2 45°+\sin^2 45°}{=} \frac{-\cos^2 45°}{\cos^2 45°} = -1$ (also compare Problem sheet 6, Problem 3); $\tan 345° = \tan(360° - 15°) = -\tan 15° = -\frac{\sin 15°}{\cos 15°} = -\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = -(2 - \sqrt{3}) = \sqrt{3} - 2$.

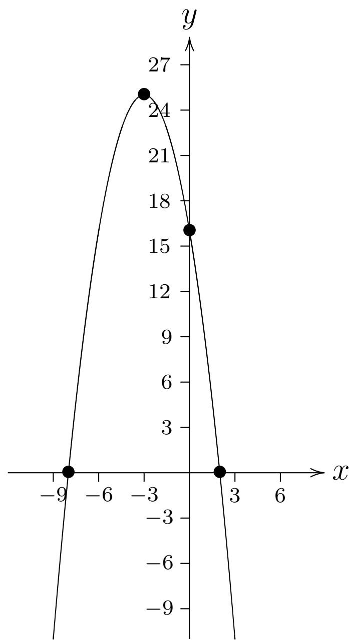
2.* Completing the square:

$$\begin{aligned} y &= 16 - 6x - x^2 = -(x^2 + 6x) + 16 \\ &= -(x^2 + 6x + 3^2 - 3^2) + 16 \\ &= -(x^2 + 6x + 9) + 9 + 16 \\ &= -(x + 3)^2 + 25. \end{aligned}$$

So, the maximum is at $(-3, 25)$.

x-intercepts: Solve $0 = 16 - 6x - x^2$ (or, with the previous calculations set $-(x + 3)^2 + 25 = 0$) to get $x = -3 \pm 5$. Thus, x-intercepts are $(2, 0)$ and $(-8, 0)$.

y-intercept: Set $x = 0$ in $y = 16 - 6x - x^2$ to get point $(0, 16)$.



If necessary, calculate additional points for sketch, e.g.:

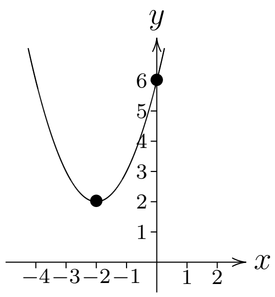
x	-10	-6	-4	-2	4	...
$y = 16 - 6x - x^2$	-24	16	24	24	-24	...

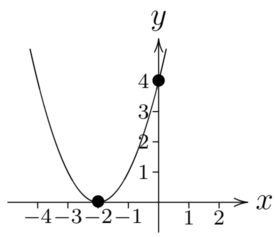
Then, also have points $(-10, -24)$, $(-6, 16)$ etc. on the curve.

Please turn over!

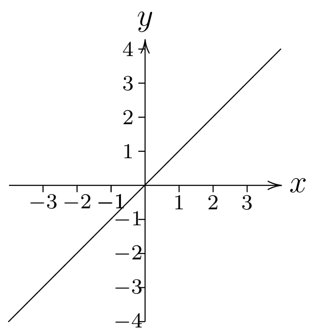
3.

- $y = x^2+4x+6$: Completing the square: $y = x^2+4x+6 = (x^2+4x+4-4)+6 = (x^2+4x+4)+2 = (x+2)^2+2$. So, minimum at $(-2, 2)$.
  No x-intercept (minimum above x-axis, no real solution of quadratic equation).
  y-intercept: Set $x = 0$ to obtain $y = 6$, thus $(0, 6)$.
  Calculate more points, e.g., $(1, 11)$, $(-1, 7)$, etc. (or translate $y = x^2$ by -2 parallel to x-axis and $+2$ parallel to y-axis).
  Sketch see below on the left.
- $y = x^2+4x+4$: Completing the square: $y = x^2+4x+4 = (x+2)^2$. So, minimum $(-2, 0)$. Minimum is also x-intercept (minimum on x-axis, exactly one solution of quadratic equation).
  y-intercept: Set $x = 0$ to obtain $y = 4$, thus $(0, 4)$.
  Sketch obtained from sketch of $y = x^2+4x+6$ by translating -2 parallel to y-axis (or $y = x^2$ by -2 parallel to x-axis).
  Sketch see below on the right.





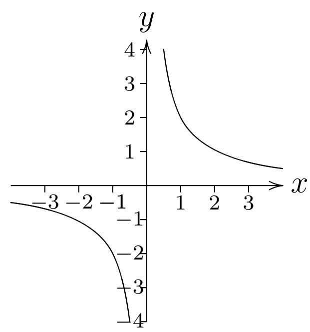
4.* (a) $y = x$: Straigth line of gradient 1 through origin (odd function).



(b) $y = \frac{2}{x}$: This is $y = \frac{1}{x}$ stretched by a factor of 2 in y-direction.
It is also an odd function: with $f(x) = \frac{2}{x}$ have $f(-x) = \frac{2}{-x} = -\frac{2}{x} = -f(x)$.

Make table

x	$\frac{1}{2}$	1	2	...
$y = \frac{2}{x}$	4	2	1	...



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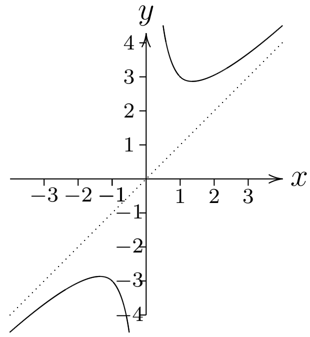
(c) $y = x + \frac{2}{x}$: x-intercepts: Solve $0 = x + \frac{2}{x}$, hence solve $0 = x^2 + 2$. No real solution, thus no x-intercept.

y-intercept: $y = x + \frac{2}{x}$ for $x = 0$ not defined ($x = 0$ is asymptote!).

Sketch (see left) obtained from (a) and (b) "by adding". Result is an odd function: Set $f(x) = x + \frac{2}{x}$, then $f(-x) = (-x) + \frac{2}{-x} = -x - \frac{2}{x} = -\left(x + \frac{2}{x}\right) = -f(x)$.

Asymptotes are $x = 0$ and $y = x$.



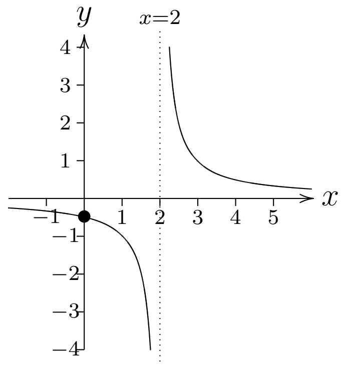
In general, the sum of two odd functions is again odd. Similarly, the sum of two even functions is again even.

5. The graph of $y = \frac{1}{x-2}$ is obtained from the graph of $y = \frac{1}{x}$ by translating by $+2$ parallel to the x-axis.

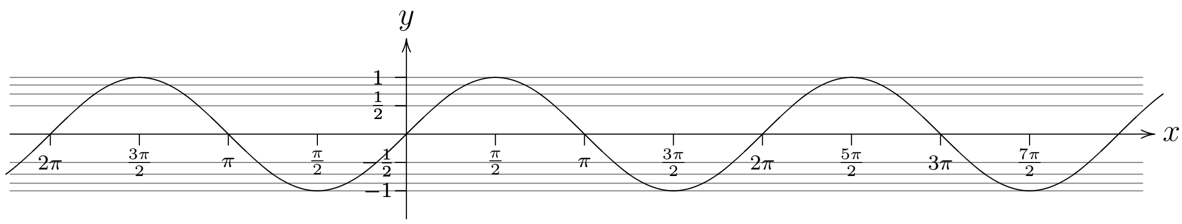
No x-intercepts, $y = 0$ is asymptote.

y-intercept: Set $x = 0$ to get $y = -\frac{1}{2}$, thus $(0, -\frac{1}{2})$ is on the curve.

Asymptotes are $y = 0$ and $x = 2$.



6. Since $\sin(-x) = -\sin x$, sine is odd.
y-intercept: Have $\sin 0 = 0$, thus y-intercept is at origin.
x-intercepts: For $0 \le x \le 2\pi$, the solutions of $\sin x = 0$ are $x = 0$ and $x = \pi$. 2π-periodicity yields that y-intercepts occur at $\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$.
Note that we already know a lot of the points on the graph, e.g., $(\frac{\pi}{6}, \frac{1}{2})$, $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$, $(\frac{\pi}{2}, 1)$ etc. (also see problem 1. on this sheet).
No asymptotes. Sketch see below (in gray: $y = \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1$).



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7. (a) $\log x = 2\log(x-1)$: Rewrite as $0 = \log(x-1)^2 - \log x = \log\frac{x^2-2x+1}{x}$. Hence, solve $1 = \frac{x^2-2x+1}{x}$, i.e., $x = x^2-2x+1$ and hence $x^2-3x+1 = 0$. Have solutions $x = \frac{3\pm\sqrt{5}}{2}$. Check $x = \frac{3+\sqrt{5}}{2}$ and $x = \frac{3-\sqrt{5}}{2}$ in equation $\log x = 2\log(x-1)$: $x = \frac{3+\sqrt{5}}{2}$ is okay, but $\log(\frac{3-\sqrt{5}}{2} - 1)$ not possible. Thus only solution is $x = \frac{3+\sqrt{5}}{2}$.

(b) $\sin^2\theta + 3\cos\theta - \frac{1}{2} = 0$: Use $\sin^2\theta = 1 - \cos^2\theta$ to obtain "quadratic" equation $1 - \cos^2\theta + 3\cos\theta - \frac{1}{2} = 0$, i.e., setting $z = \cos\theta$ we have indeed a quadratic equation: $-z^2 + 3z + \frac{1}{2} = 0$. Solutions are $z = \frac{3\pm\sqrt{11}}{2}$. Since $\frac{3+\sqrt{11}}{2} = 3.158$ (3 d.p.) is greater than 1, the possible solution is $\cos\theta = \frac{3-\sqrt{11}}{2} = -0.15831$ (5 d.p.). Hence, for $0° \leq \theta \leq 360°$, solutions are $\theta = 99.11°$ and $\theta = 260.89°$ (check!). Using 360°-periodicity, also 459.11°, 620.89° etc. and −99.11°, −260.89°, −459.11° etc. are solutions.