MA10103: Foundation Mathematics I

Solutions of Problem Sheet 6

1. $\sin\theta = -\frac{1}{2}\ :\ \theta = \frac{7}{6}\pi, \frac{11}{6}\pi;\quad \cos\theta = \frac{\sqrt{3}}{2}\ :\ \theta = \pm\frac{\pi}{6},\ \pm\frac{11}{6}\pi;\quad \tan\theta = \sqrt{3}\ :\ \theta = \frac{1}{3}\pi,\ \frac{4}{3}\pi,\ \frac{7}{3}\pi,\ \frac{10}{3}\pi.$

2*. (a) Using Pythagoras: $\sin^2\frac{2\pi}{5} = 1-\cos^2\frac{2\pi}{5}$ and $\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{3-\sqrt{5}}{8}$ one gets $\sin\frac{2\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}} = \frac{\sqrt{10+2\sqrt{5}}}{4} = 0.95106$ (5 d.p.); $\tan\frac{2\pi}{5} = \sin\frac{2\pi}{5}/\cos\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1} = 3.07768$ (5 d.p.); better solution using equation in problem 3.: $\tan^2\frac{2\pi}{5} = \frac{1}{\cos^2\frac{2\pi}{5}} - 1 = \frac{8}{3-\sqrt{5}} - 1 = \frac{24+8\sqrt{5}}{4} - 1 = \frac{20+8\sqrt{5}}{4} = 5+2\sqrt{5}$, hence $\tan\frac{2\pi}{5} = \sqrt{5+2\sqrt{5}} = 3.07768$ (5 d.p.).

(b) $\cos\theta = \frac{\sqrt{5}-1}{4}\ :\ \theta = \frac{2\pi}{5},\ \frac{8\pi}{5}.$

(c) $\sin\theta = \sin\frac{2\pi}{5}\ :\ \theta = \frac{2\pi}{5},\ \frac{3\pi}{5}.$

Note: In (b) and (c) you don't have to know the value of $\cos\frac{2\pi}{5}$ and $\sin\frac{2\pi}{5}$.

3. $\frac{1}{1+\tan^2\theta} = \frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}} \overset{(*)}{=} \frac{1}{\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}} = \frac{1}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta}{\cos^2\theta+\sin^2\theta} = \cos^2\theta.$

In step $(*)$ one uses $1 = \frac{\cos^2\theta}{\cos^2\theta}$. However, the right hand side is not defined for $\theta = \frac{\pi}{2},\ \frac{3\pi}{2}$ etc. So, for this angles a more careful analysis would be needed.

4. (a) $a = 7$, $b = 3$ and $c = 6$.
Cosine rule: $\cos A = -\frac{1}{9}$, hence $A = 96.379°$ (3 d.p.). Sine rule (using $\sin A = \frac{4\sqrt{5}}{9}$): $\sin B = \frac{3}{7}\times\frac{4\sqrt{5}}{9} = \frac{4\sqrt{5}}{21} = 0.42592$ (5 d.p.), hence $B = 25.209°$ (3 d.p.) (the solution $180° - 25.209° = 154.791°$ (3 d.p.) is not possible since $96.379° + 154.791° > 180°$). Also, $C = 180° - A - B = 58.412°$ (3 d.p.). Area: $\frac{1}{2}bc\sin A = 4\sqrt{5} = 8.944$ (3 d.p.) (preferably, the exact value $4\sqrt{5}$).

(b) $A = 20°$, $b = 5$ and $c = 4$.
Cosine rule: $a^2 = 41-40\cos 20°$, hence $a = \sqrt{41-40\cos 20°} = 1.847$ (3 d.p.). Sine rule $\sin B = \frac{5}{\sqrt{41-40\cos 20°}}\sin 20° = 0.92576$ (5 d.p.), hence $B = 67.783°$ (3 d.p.) or $B = 180° - 67.783° = 112.217°$ (3 d.p.). Sine rule: $\sin C = \frac{c}{b}\sin B = \frac{4}{5}\sin B = 0.74061$ (5 d.p.), hence $C = 47.783°$ (3 d.p.) or $C = 180° - 47.783° = 132.217°$ (3 d.p.). Only possibility with $A+B+C = 180°$ is $B = 112.217°$ (3 d.p.) and $C = 47.783°$ (3 d.p.). The area is $\frac{1}{2}bc\sin A = 10\sin 20° = 3.420$ (3 d.p.).

Please turn over!

(c) $A = 50°$, $a = 7$ and $b = 4$.
Sine rule: $\sin B = \frac{4}{7} \sin 50° = 0.43774$ (5 d.p.), hence $B = 25.960°$ (3 d.p.) (other value $154.040°$ (3 d.p.) is not possible). Then $C = 180° - A - B = 104.040°$ (3 d.p.) and, using the sine rule, $c = a\frac{\sin C}{\sin A} = 8.865$ (3 d.p.). Area: $\frac{1}{2}bc \sin A = 13.582$ (3 d.p.).

(d) $A = 105°$, $B = 44°$ and $a = 5$.
$C = 180° - A - B = 31°$, then sine rule $b = 5\frac{\sin 44°}{\sin 105°} = 3.596$ (3 d.p.) and $c = 5\frac{\sin 31°}{\sin 105°} = 2.666$ (3 d.p.). Area is $\frac{1}{2}ab \sin C = \frac{25}{2} \times \frac{\sin 44° \times \sin 31°}{\sin 105°} = 4.630$ (3 d.p.).

5*. $A = 50°$, $a = 4$ and $b = 5$.
Sine rule $\sin B = \frac{5}{4} \sin 50° = 0.95756$ (5 d.p.), hence $B_1 = 73.247°$ (3 d.p.) and $B_2 = 106.753°$ (3 d.p.). Thus, $C_1 = 56.753°$ and $C_2 = 23.247°$ (3 d.p.). Sine rule $c_1 = \frac{4}{\sin 50°} \sin C_1 = 4.367$ (3 d.p.) and $c_2 = 2.061$ (3 d.p.).
One triangle contains a, b, c_1, A, B_1, C_1, the other a, b, c_2, A, B_2, C_2.

6. From problem 4. and examples in lecture!

In the following cases there is only **one** possible solution/triangle!

given	unknowns calculated using
all three sides	angles by cosine rule *or* 1st angle by cosine rule, 2nd by sine rule, 3rd by sum
two sides and enclosed angle	3rd side by cosine rule, one angle by sine rule, the other by sum (if possible) *or* 3rd side by cosine rule, both angles by sine rule (exclude possibilities using sum of angles) *or* 3rd side and one angle by cosine rule, last angle by sum
two sides and angle opposing the longer side	angle opposing shorter side by sine rule, 3rd angle by sum, 3rd side by sine rule
one side and two angles	3rd angle by sum, sides by sine rule

Cosine rule only in the first step needed if no pair a, A or b, B or c, C given.
Use $A + B + C = 180°$ for 3rd angle (to be sure that sum **is** $180°$), if possible.
Using the sine rule, you might get two possible angles, but in the cases here, only one of them is actually possible. So one of the possibilities can be excluded either simply by calculating the sum of the angles, or after using the sine rule for the third angle and calculating the sum.