MA10103: Foundation Mathematics I

Solutions of Problem Sheet 5

1. $x^2 + x - 6 = 0 \ : \ x = \frac{-1\pm5}{2} = 2$ or -3; $2x^2 + 1 = -3x \ : \ x = \frac{-3\pm1}{4} = -1$ or $-\frac{1}{2}$; $6x^2 = 7x + 3 \ : \ x = \frac{7\pm11}{12} = \frac{3}{2}$ or $-\frac{1}{3}$; $-4x^2 + 12x - 9 = 0 \ : \ x = \frac{-12\pm0}{-8} = \frac{3}{2}$ (only one solution); $x^2 - 3x - 3 = 0 \ : \ x = \frac{3\pm\sqrt{21}}{2}$.

2. $x^2 + bx + c = 0$ has exactly one solution if $b^2 = 4 \times c$.
$2x^2 + bx = -9$ has one solution if $b^2 - 72 = 0$, i.e., if $b = \pm6\sqrt{2}$. It has two solutions if $b^2 - 72 > 0$, i.e., if $b > 6\sqrt{2}$ or if $b < -6\sqrt{2}$.

3. $x^2 + 5x + c = 0$ has no real solutions if $25 - 4c < 0$, i.e., if $c > \frac{25}{4} = 6.25$. So, for $c = 7$ or $c = 10$ or etc. the equation has no real solutions.

4*. **(a)** $x^2 + 3x + 1 = 0 \ : \ x = \frac{-3\pm\sqrt{5}}{2}$, i.e., $x = \frac{-3+\sqrt{5}}{2} = -0.382$ (3 d.p.) or $x = \frac{-3-\sqrt{5}}{2} = -2.618$ (3 d.p.);
$x^2 - x - 6 = 0 \ : \ x = \frac{1\pm5}{2} = 3$ or -2

(b) $4^x + 3 \times 2^x + 1 = 0 \ :$ set $y = 2^x$, then $y^2 + 3y + 1 = 0$. This is the first equation in (a) and has solutions $y = \frac{-3\pm\sqrt{5}}{2}$. However, these solutions for y are negative, while 2^x is positive for any real number x. Thus, there is no solution for $4^x + 3 \times 2^x + 1 = 0$.

$9^x - 3^x - 6 = 0 \ :$ set $y = 3^x$, then $y^2 - 3y - 6 = 0$. This is the second equation in (a) and has solutions $y = 3$ and $y = -2$. As before, 3^x is positive for any real number x, so we only have to consider the solution $y = 3$. Hence, $3^x = 3$ and therefore $x = 1$. The result is: $9^x - 3^x - 6 = 0$ has one solution, namely $x = 1$.

(c) $x^4 + 3x^2 + 1 = 0 \ :$ set $y = x^2$, then $y^2 + 3y + 1 = 0$. This is the first equation in (a) and has solutions $y = \frac{-3\pm\sqrt{5}}{2}$. Both are negative, while the square of any real number x is positive. Thus, there are no real solutions for this equation.

$x^4 - x^2 - 6 = 0 \ :$ set $y = x^2$, then $y^2 - 3y - 6 = 0$. This is the second equation in (a) and has solutions $y = 3$ and $y = -2$. Again, we only have to care about the positive solution. Therefore, we have to solve $x^2 = 3$ which has solutions $x = \pm\sqrt{3}$. So, $x^4 - x^2 - 6 = 0$ has the two solutions $x = \sqrt{3}$ and $x = -\sqrt{3}$.

Please turn over!

5*. (a) $x - 2y = -7$ and $-2x + 3y = 9$: $x = 3$ and $y = 5$.

(b) $3x + 2y = -6$ and $x + y = 1$: $x = -8$ and $y = 9$.

(c) $xy = 1$ and $x + y - 3 = 0$: setting $y = 3 - x$, solutions of $-x^2 + 3x - 1 = 0$ are $x = \frac{3\pm\sqrt{5}}{2}$. Get solutions $x = \frac{3+\sqrt{5}}{2}$, $y = \frac{3-\sqrt{5}}{2}$ and $x = \frac{3-\sqrt{5}}{2}$, $y = \frac{3+\sqrt{5}}{2}$.

(d) $x + y = 2$ and $y^2 - x^2 = 8$: setting $y = 2 - x$, solution of $4 - 4x = 8$ is $x = -1$. But then $y = 3$, and the solution is $x = -1$ and $y = 3$.

(e) $2x - y = 2$ and $x^2 - y = 5$: setting $y = 2x - 2$, solutions of $x^2 - 2x - 3 = 0$ are $x = \frac{2\pm4}{2} = 3$ or -1. So, solutions are $x = 3$, $y = 4$ and $x = -1$, $y = -4$.

6. Set $z = \log_2 x$.
Solve $z + 3y = 5$ and $z^2 + y(y-1) = 3$: setting $z = 5 - 3y$, one obtains the equation $3 = (5-3y)^2 + y(y-1) = 25 - 31y + 10y^2$ and hence $10y^2 - 31y + 22 = 0$. This has solutions $y = \frac{31\pm9}{20} = 2$ or 1.1. Hence, $z = -1$, $y = 2$ and $z = 1.7$ and $y = 1.1$ are solutions of $z + 3y = 5$ and $z^2 + y(y-1) = 3$.
Using $z = \log_2 x$ and therefore $2^z = x$, one therefore gets: $x = \frac{1}{2}$, $y = 2$ and $x = 2^{1.7}$, $y = 1.1$ (one has $2^{1.7} = 3.249$ (3 d.p.)).

7*. $\sin\theta = 1 \ : \ \theta = \frac{\pi}{2}$; $\cos\theta = -1 \ : \ \theta = \pi$; $\sin\theta = 1/2 \ : \ \theta = \frac{\pi}{6}, \frac{5\pi}{6}, 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$ and $2\pi + \frac{5\pi}{6} = \frac{17\pi}{6}$; $\tan\theta = 1 \ : \ \theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$ and $\frac{13\pi}{4}$.

8. (a) $20° = \frac{\pi}{9}$ radians; $70° = \frac{7\pi}{18}$ radians; $105° = \frac{7\pi}{12}$ radians;
$288° = \frac{8\pi}{5}$ radians; $348° = \frac{29\pi}{15}$ radians.

(b) $\frac{1}{3}\pi$ radians $= 60°$; $\frac{6}{5}\pi$ radians $= 216°$; $\frac{15}{4}\pi$ radians $= 675°$;
$\frac{7}{8}\pi$ radians $= 157.5°$; $\frac{5}{3}\pi$ radians $= 300°$.

(c) $\sin 22° = 0.3746$ (4 d.p.); $\cos\left(\frac{3}{7}\pi\right) = 0.2225$ (4 d.p.);
$\sin 108° = 0.9511$ (4 d.p.); $\sin\left(\frac{2\pi}{9}\right) = 0.6427$ (4 d.p.);
$\tan\left(\frac{2}{5}\pi\right) = 3.0777$ (4 d.p.).