MA10103: Foundation Mathematics I

Solutions of Problem Sheet 11

Problems 1. to 5.

1.* $f(x) = \mathrm{e}^{-3x}$: $f'(x) = -3\,\mathrm{e}^{-3x}$ (chain rule); $f(x) = \sin x\,\cos x$: $f'(x) = \cos x\,\cos x + \sin x\,(-\sin x) = \cos^2 x - \sin^2 x$ (product rule); $f(x) = x^2 \log x$: $f'(x) = 2x \log x + x^2\,\frac{1}{x} = x + 2x \log x$ (product rule); $f(x) = \sin(x^3+2)$: $f'(x) = 3\,x^2\;\cos(x^3+2)$ (chain rule); $f(x) = \ln(\sin(x^3+2))$: $f'(x) = \frac{1}{\sin(x^3+2)} \times \frac{\mathrm{d}}{\mathrm{d}x}\sin(x^3+2) = 3\,x^2\,\frac{\cos(x^3+2)}{\sin(x^3+2)} = 3\,x^2\;\cot(x^3+2)$.

2.*
- $y = \ln(1+x^2)$: Chain rule yields $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{1+x^2}$. Stationary point if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, this is the case for $x = 0$, so stationary point has y-coordinate $\ln(1+0^2) = 0$. Second derivative using quotient rule is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2-2x^2}{(1+x^2)^2}$. At $x = 0$, second derivative is 2 and thus positive, wherefore the stationary point $(0,0)$ is a minimum.
- $y = (x+1)^2$: Derivative is $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x+2$, so stationary point at $x = -1$ (y-coordinate is $(-1+1)^2 = 0$). Second derivative is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$, so stationary point at $(-1,0)$ is a minimum.
- $y = x - \mathrm{e}^x$: Derivative is $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \mathrm{e}^x$; solving $1 - \mathrm{e}^x = 0$ yields $x = 0$. Second derivative is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\mathrm{e}^x$, so the stationary point at $(0, 0 - \mathrm{e}^0) = (0,-1)$ is a maximum (since $-\mathrm{e}^0 = -1 < 0$).
- $y = x^3 - 3x^2 - 9x + 1$: Derivative is $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x - 9 = 3\,(x-3)(x+1)$, so stationary points at $x = 3$ (with y-coordinate -26) and at $x = -1$ (with y-coordinate 6). Second derivative is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 6$, thus stationary point at $(-1,6)$ is a (local) maximum $(6 \times (-1) - 6 = -12 < 0)$, while stationary point at $(3,-26)$ is a (local) minimum $(6 \times 3 - 6 = 12 > 0)$.

3.* Assuming that the cardboard is a square of area $1\,\mathrm{m}^2$, its length and width are both $1\,\mathrm{m}$ (as an additional exercise, redo this example if the cardboard is not a square, e.g., if the cardboard is twice as long as wide, but still has area $1\,\mathrm{m}^2$).

The folded box has length $1-2x$, width $1-2x$ and height x. Thus the volume is

$$V = (1-2x)(1-2x)x = 4x^3 - 4x^2 + x.$$

Note that for a real, existing box, we must have $0 < x < \frac{1}{2}$ (all numbers in metres). To find the maximal possible volume, we differentiate:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 8x + 1.$$

Please turn over!

Stationary value for $\frac{dV}{dx} = 0$, so have to solve $12x^2 - 8x + 1 = 0$. This quadratic equation has solutions $x = \frac{1}{6}$ and $x = \frac{1}{2}$. Note that at $x = \frac{1}{2}$ (and similar for $x = 0$), the volume vanishes. At $x = \frac{1}{6}$ m, the volume is $V = \frac{2}{27}\,\mathrm{m}^3$. Using the second derivative $\frac{d^2V}{dx^2} = 24x - 8$ at $x = \frac{1}{6}$ (which yields -4, a negative number), one confirms that this is indeed the maximal value.

4.* Note: The distance of the particle to the origin is given by $\sqrt{x^2 + y^2}$. Moreover, if $\sqrt{x^2 + y^2}$ is minimal, so is its square $x^2 + y^2$. Here, we calculate the stationary points of the square of the distance.

(a) Square of the distance is $x^2 + y^2 = \frac{1}{2}\cos^2 t + 2\sin^2 t$. Using the product or the chain rule shows that $\frac{d}{dt}\cos^2 t = -2\cos t\,\sin t$ and $\frac{d}{dt}\sin^2 t = 2\cos t\,\sin t$, wherefore

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}\left(\frac{1}{2}\cos^2 t + 2\sin^2 t\right)$$
$$= \frac{1}{2}(-2)\cos t\,\sin t + 2 \times 2\cos t\,\sin t = 3\,\sin t\,\cos t.$$

(b) Stationary points where derivative vanishes, i.e., at times t such that $3\,\sin t\,\cos t = 0$. Thus, we are looking for t (where $0 \le t \le 2\pi$) such that either $\sin t = 0$ or $\cos t = 0$.

- $\cos t = 0$: This is the case for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. In these both cases, the distance to the origin is $\sqrt{\frac{1}{2}\,0^2 + 2\,(\pm 1)^2} = \sqrt{2}$.
- $\sin t = 0$: This is the case for $t = 0$, $t = \pi$ and $t = 2\pi$. In all these cases, the distance to the origin is $\sqrt{\frac{1}{2}\,(\pm 1)^2 + 2\,0^2} = \sqrt{\frac{1}{2}}$.

So, the closest the particle comes to the origin is $\sqrt{\frac{1}{2}}$ (at $t = 0, \pi, 2\pi$; the maximal distance is $\sqrt{2}$, attained at times $t = \frac{\pi}{2}, \frac{3\pi}{2}$). In fact, the particle travels on an ellipse around the origin (with semiaxes of length $\sqrt{2}$ and $\sqrt{\frac{1}{2}}$).

5.* Volume of a spherical balloon: $V = \frac{4}{3}\pi r^3$. So, as function of time t, we have $V(t) = \frac{4}{3}\pi\,(r(t))^3$.

Volume increase (using chain rule): $\frac{dV}{dt} = \frac{4}{3}\pi \times 3\,(r(t))^2\,\frac{d}{dt}r(t) = 4\pi\,(r(t))^2\,r'(t)$. At the time of observation, we have $r = 10\,\mathrm{cm}$ and $\frac{dr}{dt} = 3\,\mathrm{cm/min}$; so at that time, the volume increase (i.e., how fast the air is pumped in) is

$$\frac{dV}{dt} = 4\pi \times (10\,\mathrm{cm})^2 \times 3\,\mathrm{cm/min} = 1200\pi\,\mathrm{cm}^3/\mathrm{min}.$$