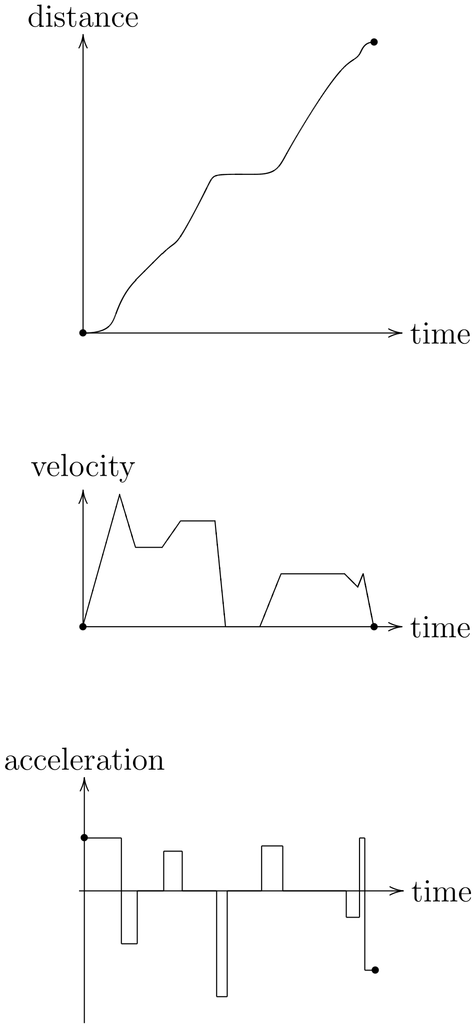
MA10103: Foundation Mathematics I

Solutions of Problem Sheet 10

1. See right column (only a suggestion).



2. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x+1} \overset{\text{quotient rule}}{=} \frac{-1}{(x+1)^2}$;

$\frac{\mathrm{d}}{\mathrm{d}x}\left((\sin x)\,(x+2)\right) \overset{\text{product rule}}{=} (\cos x)\,(x+2) + \sin x$;

$\frac{\mathrm{d}}{\mathrm{d}x}\sin(2\,x+3) \overset{\text{chain rule}}{=} \cos(2x+3)\times 2$;

$\frac{\mathrm{d}}{\mathrm{d}x}\left((x^3+2)^6\right) \overset{\text{chain rule}}{=} 6\,(x^3\;+\;2)^5\;\times\;(3x^2)\;=\;18x^2\,(x^3+2)^5$;

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x+(3-x)^5\right) = 1 + \frac{\mathrm{d}}{\mathrm{d}x}\left((3-x)^5\right) \overset{\text{chain rule}}{=} 1+5\,(3-x)^4\times(-1) = 1-5\,(3-x)^4$;

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos(2\,x)}{x^2}\right) \overset{\text{quotient rule}}{=} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}\cos(2x)\right)x^2-\cos(2x)\times 2x}{(x^2)^2} \overset{\text{chain rule}}{=} \frac{-2\sin(2x)\,x^2-\cos(2x)\times 2x}{x^4} = \frac{-2x\sin(2x)-2\cos(2x)}{x^3}$;

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\,\cos(3\,x)\right) \overset{\text{product rule}}{=} 3x^2\,\cos(3x)\;+\;x^3\;\times\;\frac{\mathrm{d}}{\mathrm{d}x}\cos(3\,x) \overset{\text{chain rule}}{=} 3x^2\,\cos(3\,x)+x^3\,(-3\sin(3\,x)) = 3x^2\,(\cos(3\,x)-x\sin(3\,x))$;

$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+2\,x} \overset{\text{chain rule}}{=} \frac{1}{2\sqrt{1+2\,x}}\,2 = \frac{1}{\sqrt{1+2\,x}}$;

$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+x^2} \overset{\text{chain rule}}{=} \frac{1}{2\sqrt{1+x^2}}\,2x = \frac{x}{\sqrt{1+x^2}}$.

3.* $f(x) = (x^2+1)^{10}$: Using the chain rule with $g(x) = x^{10}$ and $h(x) = x^2+1$ (wherefore $g'(x) = 10x^9$, $h'(x) = 2x$ and furthermore $f'(x) = g'(h(x))\times h'(x)$) one obtains

$$f'(x) = 10\,(x^2+1)^9\,2x = 20x(x^2+1)^9.$$

Moreover, $f'(1) = 20\times1\times(1^2+1)^9 = 20\times2^9 = 10240$.

4.* $g(x) = \sin\left((x^2+1)^9\right)$: Using the chain rule with $f(x) = \sin x$ and $h(x) = (x^2+1)^9$ (wherefore $f'(x) = \cos x$ and – compare previous question – also $h'(x) = 18x(x^2+1)^8$) and thus $g(x) = f(h(x))$ one obtains:

$$g'(x) = \cos\left((x^2+1)^9\right)\times 18x\,(x^2+1)^8.$$

Please turn over!

5. (a) Completing the square: $y = 3\,x^2 + 4x - 3 = 3\,(x^2 + \frac{4}{3}x - 1) =$
$3\left((x + \frac{2}{3})^2 - \left(\frac{2}{3}\right)^2 - 1\right) = 3(x + \frac{2}{3})^2 - \frac{13}{3}$.
So, the lowest point is $(-\frac{2}{3}, -\frac{13}{3})$.

(b) Derivative: $\frac{\mathrm{d}y}{\mathrm{d}x} = 6\,x + 4$.
So, stationary point for $6\,x + 4 = 0$, i.e., for $x = -\frac{2}{3}$.
Calculating corresponding y-value: $y = 3\,(-\frac{2}{3})^2 + 4 \times (-\frac{2}{3}) - 3 = -\frac{13}{3}$.
Again, the lowest point is $(-\frac{2}{3}, -\frac{13}{3})$.

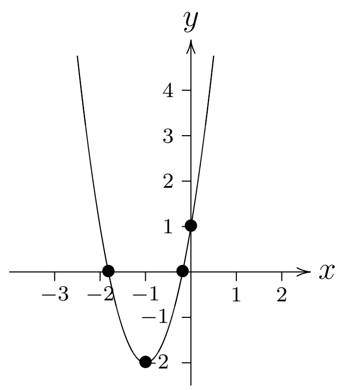
6.* (a) $f'(x) = 3x^2 + 6x + 1$.

y-intercept of f': $(0, 1)$.

x-intercept of f': Solve $3x^2 + 6x + 1 = 0$ with solution formula, so $x = \frac{-6 \pm \sqrt{36-12}}{6} = -1 \pm \sqrt{\frac{2}{3}}$. Thus, x-intercepts are $(-1 - \sqrt{\frac{2}{3}}, 0)$ and at $(-1 + \sqrt{\frac{2}{3}}, 0)$.

Stationary point of f': Calculate derivative of f'. This is $f''(x) = 6x + 6$. Stationary point when $6x + 6 = 0$, so at $x = -1$. We have a minimum of $f'(x)$ at $(-1, -2)$.

Sketch of $f'(x) = 3x^2 + 6x + 1$:



(b) x-coordinates of the stationary points of $y = f(x)$ are the x-intercepts of $f'(x)$. So, stationary points are at $x = -1 \pm \sqrt{\frac{2}{3}}$, i.e., stationary points are $(-1 - \sqrt{\frac{2}{3}}, f(-1 - \sqrt{\frac{2}{3}})) \approx (-1.8165, 2.0887)$ and $(-1 + \sqrt{\frac{2}{3}}, f(-1 + \sqrt{\frac{2}{3}})) \approx (-0.1835, -0.0887)$.
$f(x)$ is increasing if $f'(x)$ is positive, thus it is increasing for $x < -1 - \sqrt{\frac{2}{3}}$ and for $x > -1 + \sqrt{\frac{2}{3}}$.
It is decreasing if $f'(x)$ is negative, so for $-1 - \sqrt{\frac{2}{3}} < x < -1 + \sqrt{\frac{2}{3}}$.
So, stationary point at $x = -1 - \sqrt{\frac{2}{3}}$ is maximum, while the one at $x = -1 + \sqrt{\frac{2}{3}}$ is a minimum.

(c) y-intercept at origin $(0, 0)$.

x	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1
$f(x)$	-3	0.625	2	1.875	1	0.125	0	1.375	5

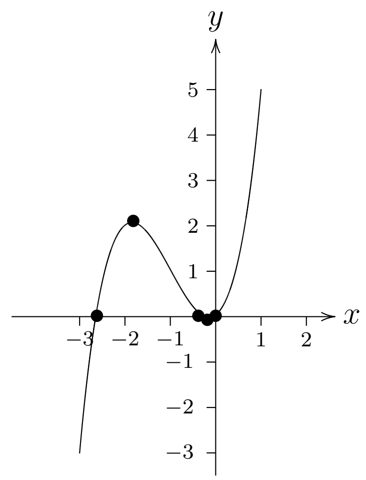
Please turn over!

Factorisation: $x^3+3\,x^2+x = x\,(x^2+3\,x+1)$. Find factorisation of $x^2+3\,x+1$ using the solution formula for $x^2+3\,x+1=0$. This yields $x=\frac{-3\pm\sqrt{5}}{2}$. Thus, factorisation is

$$x^3+3\,x^2+x =$$
$$x\left(x-\frac{-3+\sqrt{5}}{2}\right)\left(x-\frac{-3-\sqrt{5}}{2}\right).$$

So, x-intercepts are at $x=\frac{-3-\sqrt{5}}{2}\approx -2.618$, $x=\frac{-3+\sqrt{5}}{2}\approx -0.382$ and at $x=0$.

Sketch of $f(x)=x^3+3\,x^2+x$:



7. $y=\tan x$: Aymptotes at $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\pm\frac{7\pi}{2},\ldots$ (π-periodic).
x-intercepts at $x=0,\pm\pi,\pm2\pi,\pm3\pi,\ldots$ (π-periodic).